System of linear equations

A system of m equations with n unknowns (an $m \times n$ system) can be written in the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

where a_{ij} is called the coefficient of the unknown x_j, and b_i is called the constant (or free term) of the ith equation.

The system can be rewritten as a matrix equation of the form $AX = B$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$$

are called the matrix of coefficients, the matrix of unknowns and the matrix of constants (or free terms), respectively.

Moreover, if the coefficient matrix is augmented by the constant matrix, then we get the augmented matrix of the system:

$$\overline{A} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right).$$

The linear system must have

(A) exactly one solution (then it is called consistent and independent);

(B) infinitely many solutions (then it is called consistent and dependent);

(C) no solution (then it is called inconsistent).

There are no other possibilities.

Remark 1 *"Independent" means that each equation gives new information. Otherwise they are "dependent".*

These three cases have a geometric illustration when the system consists of two equations with two unknowns.

Example 1 *Solve the following linear systems:*

(A) $\begin{cases} 3x_1 + x_2 = 1 \\ 5x_1 + 3x_2 = -1 \end{cases}$

(B) $\begin{cases} 3x_1 + x_2 = 1 \\ -6x_1 - 2x_2 = -2 \end{cases}$

(C) $\begin{cases} 3x_1 + x_2 = 1 \\ 3x_1 + x_2 = -1 \end{cases}$

Solution 1 *(A) Let us use the substitution technique:*

$$x_2 = -3x_1 + 1$$

$$3(-3x_1 + 1) + 5x_1 = -1$$

$$-9x_1 + 3 + 5x_1 = -1$$

$$-4x_1 = -4$$

$$x_1 = 1$$

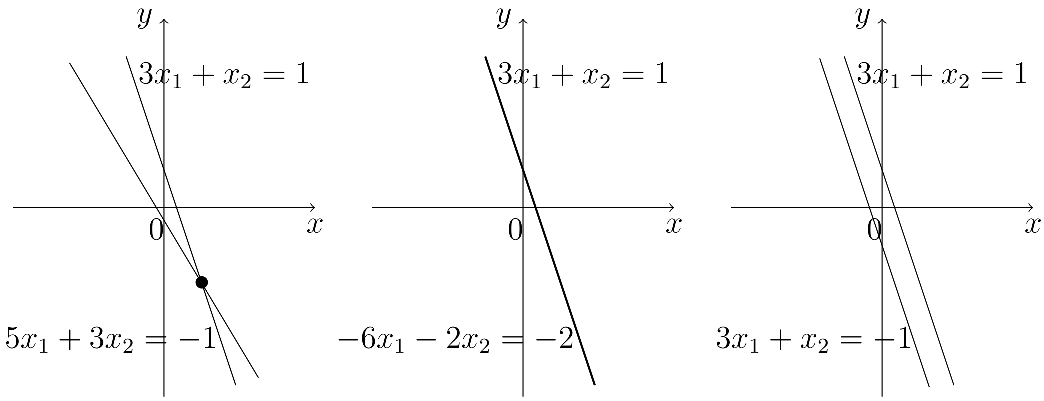
$$x_2 = -2$$

The graphs of both equations are straight lines $x_2 = -3x_1 + 1$ *and* $x_2 = -\frac{5}{3}x_1 - \frac{1}{3}$*. They are shown in Figure 1 (A). They intersect at the point (1;-2), the solution of system.*

(B) Those equations are "dependent", because they are really the same equations. So, the second equation gives no new information. They have the same graph, as shown in Figure 1 (B). Thus, all points of the straight line $x_2 = -3x_1 + 1$ *satisfy the system. It means that the system has infinitely many solutions, namely,* x_1 *is any real number, while* $x_2 = -3x_1 + 1$*.*

(C) The graphs of these equations are parallel lines $x_2 = -3x_1 + 1$ *and* $x_2 = -3x_1 - 1$ *(each has slope* -3*), as shown in Figure 1 (C). Therefore, the system has no solution.*

However, it is not necessary to draw the graphs to discover this fact. If we subtract the second equation from the first, we get the false statement: $0 = 2$. *It means that the system is inconsistent.*



(A) Lines intersect at one point: exactly one solution *(A) Lines coincide: infinitely many solutions* *(A) Lines are parallel: no solution*

Figure 1

Solving a system of linear equations using Gaussian elimination

Theorem 1 *If augmented matrices of two systems of linear equations are row equivalent, then these systems have the same solution.*

Step 1 (forward elimination). Step-by-step reduction of the augmented matrix of the given system into an equivalent echelon form matrix.
Step 2 (backward elimination). Step-by-step back-substitution to find a solution of the simpler system constructed from the obtained equivalent echelon form matrix.

Remark 2 *From Step 2 it is obvious that a system has a solution if and only if an echelon form of its augmented matrix does not have a row of the form* $(0, 0, 0, ..., b)$ *with* $b \neq 0$.

Example 2 *Solve the system* $\begin{cases} x_1 + x_2 - 2x_3 + 4x_4 = 5 \\ 2x_1 + 2x_2 - 3x_3 + x_4 = 3 \\ 3x_1 + 3x_2 - 4x_3 - 2x_4 = 1 \end{cases}$.

Solution 2

$$\overline{A} = \left(\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 2 & 2 & -3 & 1 & 3 \\ 3 & 3 & -4 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 2 & -14 & -14 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

That is the given system has the same solution as the system:

$$\begin{cases} x_1 + x_2 - 2x_3 + 4x_4 = 5 \\ \qquad\qquad\quad x_3 - 7x_4 = -7 \end{cases}.$$

From the last system by back-substitution we have that $x_3 = -7 + 7x_4$ *and* $x_1 = -9 - x_2 + 10x_4$*. Thus, if* a *and* b *are any real numbers, then*

$$x_1 = -9 - a + 10b$$

$$x_2 = a$$

$$x_3 = -7 + 7a$$

$$x_4 = b$$

are solutions of the given system.

Example 3 *Solve the system* $\begin{cases} x_1 + x_2 - 2x_3 + 3x_4 = 4 \\ 2x_1 + 3x_2 + 3x_3 - x_4 = 3 \\ 5x_1 + 7x_2 + 4x_3 + x_4 = 5 \end{cases}.$

Solution 3

$$\overline{A} = \left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & 1 & 5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 2 & 14 & -14 & -15 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right).$$

That is the $0 \cdot x_4 = -5$*. Thus, the system has no solution.*

Example 4 *Solve the system* $\begin{cases} x_1 + 2x_2 + x_3 = 3 \\ 2x_1 + 5x_2 - x_3 = -4 \\ 3x_1 - 2x_2 - x_3 = 5 \end{cases}.$

Solution 4

$$\overline{A} = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 5 & -1 & -4 \\ 3 & -2 & -1 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & -8 & -4 & -4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & -28 & -84 \end{array}\right).$$

That is the given system has the same solution as the system:

$$\begin{cases} x_1 + 2x_2 + x_3 = 3 \\ \qquad x_2 - 3x_3 = -10 \\ \qquad\quad -28x_3 = -84 \end{cases}.$$

From the last system by back-substitution we get that

$$x_1 = 2$$
$$x_2 = -1$$
$$x_3 = 3$$

is a solution of the given system.

Cramer's rule for square linear systems

Let us consider the general system of linear equations when $m = n$. In this case it is called the square system of linear equations. Since this system can be represented as a matrix equation $AX = B$, we have $X = A^{-1}B$ if $\det A \neq 0$.

Theorem 2 *Any square system of linear equations has a unique solution if and only if the determinant of its coefficients matrix is not zero.*

Suppose that $C = (c_{ij})_{n\times n}$ with $c_{ij} = (-1)^{i+j}M_{ij}$. We know that $A^{-1} = \frac{1}{\det A}C^T$. Thus,

$$X = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} c_{11} & c_{12} & ... & c_{1n} \\ c_{21} & c_{22} & ... & c_{21} \\ ... & ... & ... & ... \\ c_{n1} & c_{n2} & ... & c_{nn} \end{pmatrix}^T \begin{pmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{pmatrix}$$

$$= \frac{1}{\det A} \begin{pmatrix} c_{11} & c_{21} & ... & c_{n1} \\ c_{12} & c_{22} & ... & c_{n2} \\ ... & ... & ... & ... \\ c_{1n} & c_{2n} & ... & c_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{pmatrix} = \begin{pmatrix} c_{11}b_1 + c_{21}b_2 + \cdots + c_{n1}b_n \\ c_{12}b_1 + c_{22}b_2 + \cdots + c_{n2}b_n \\ ... \\ c_{1n}b_1 + c_{2n}b_2 + \cdots + c_{nn}b_n \end{pmatrix}.$$

Hence,

$$x_1 = \frac{c_{11}b_1 + c_{21}b_2 + \cdots + c_{n1}b_n}{\det A}$$
$$x_2 = \frac{c_{12}b_1 + c_{22}b_2 + \cdots + c_{n2}b_n}{\det A}$$
$$\cdots$$
$$x_n = \frac{c_{1n}b_1 + c_{2n}b_2 + \cdots + c_{nn}b_n}{\det A}$$

Let

$$\det A_1 = \begin{vmatrix} b_1 & a_{12} & ... & a_{1n} \\ b_2 & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ b_n & a_{n2} & ... & a_{nn} \end{vmatrix}.$$

It means that the matrix A_1 is a matrix formed from A by replacing its first column by the column of free terms. If we use "expansion by minors" with respect to this first column, we get

$$\det A_1 = \sum_{i=1}^{n} (-1)^{i+1} b_i M_{i1} = c_{11}b_1 + c_{21}b_2 + \cdots + c_{n1}b_n.$$

Therefore,

$$x_1 = \frac{\det A_1}{\det A}.$$

Similarly, we can write a general rule for all unknowns:

$$x_i = \frac{\det A_i}{\det A},$$

where A_i is the matrix formed by replacing the ith column of A with the column of free terms. This rule is known as *Cramer's rule.*

Example 5 *Solve the following system:* $\begin{cases} 3x_1 + x_2 = 1 \\ 5x_1 + 3x_2 = -1 \end{cases}$.

Solution 5

$$\det A = \begin{vmatrix} 3 & 1 \\ 5 & 3 \end{vmatrix} = 3 \cdot 3 - 1 \cdot 5 = 4$$

$$\det A_1 = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 1 \cdot 3 - 1 \cdot (-1) = 4$$

$$\det A_2 = \begin{vmatrix} 3 & 1 \\ 5 & -1 \end{vmatrix} = 3 \cdot (-1) - 1 \cdot 5 = -8$$

$$x_1 = \frac{\det A_1}{\det A} = \frac{4}{4} = 1$$

$$x_2 = \frac{\det A_2}{\det A} = \frac{-8}{4} = -2$$