Determinants

Determinant is a number associated with any square matrix A and denoted by $\det A$. If vertical lines are around a matrix, it means determinant. Thus,

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

It can be calculated as follows:

1. if $n = 1$, then $\det A = a_{11}$;

2. if $n > 1$, then

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \text{ for any fixed } i$$

or

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \text{ for any fixed } j,$$

where M_{ij} is a determinant called a minor that results by eliminating the ith row and jth column from A. This calculation is called "Expansion by minors".

Thus, in particular, if $n = 2$ and $i = 1$ we have

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \sum_{j=1}^{2} (-1)^{1+j} a_{1j} M_{1j} = (-1)^{1+1} a_{11} M_{11} + (-1)^{1+2} a_{12} M_{12}$$

$$= a_{11} a_{22} - a_{12} a_{21}.$$

The obtained formula is used to calculate the determinants of any 2×2 matrices. So, it is the product of elements on the principal diagonal minus the product of elements on the nonprincipal diagonal.

Similarly, if $n = 3$ and $i = 1$ we have

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{j=1}^{3} (-1)^{1+j} a_{1j} M_{1j}$$

$$= (-1)^{1+1} a_{11} M_{11} + (-1)^{1+2} a_{12} M_{12} + (-1)^{1+3} a_{13} M_{13}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}).$$

This formula is used to calculate the determinants of any 3×3 matrices. To remember this formula we need to extend the determinant's grid by rewriting the first two columns:

$$\left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right|.$$

There are three down-diagonals: the principal diagonal with the elements a_{11}, a_{22}, a_{33}, and two diagonals parallel to principal with the elements a_{12}, a_{23}, a_{31} and a_{13}, a_{21}, a_{32}, respectively. If we sum the products of elements on each of these down-diagonals, we get the first bracket of the formula. Similarly, there are three up-diagonals: the nonprincipal diagonal with the elements a_{13}, a_{22}, a_{31}, and two diagonals parallel to nonprincipal with the elements a_{11}, a_{23}, a_{32} and a_{12}, a_{21}, a_{33}, respectively. If we sum the products of elements on each of these up-diagonals, we get the second bracket of the formula.

Example 1 *Find* $\det A$ *if* $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$.

Solution 1

$$\det A = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1 \cdot (-1) - 2 \cdot 3 = -1 - 6 = -7.$$

Example 2 *Find* $\det A$ *if* $A = \begin{pmatrix} 5 & -2 & 1 \\ 3 & 1 & -4 \\ 6 & 0 & -3 \end{pmatrix}$.

Solution 2

$$\det A = \left|\begin{array}{ccc|cc} 5 & -2 & 1 & 5 & -2 \\ 3 & 1 & -4 & 3 & 1 \\ 6 & 0 & -3 & 6 & 0 \end{array}\right|$$

$$= (5 \cdot 1 \cdot (-3) + (-2) \cdot (-4) \cdot 6 + 1 \cdot 3 \cdot 0) - (1 \cdot 1 \cdot 6 + 5 \cdot (-4) \cdot 0 + (-2) \cdot 3 \cdot (-3))$$

$$= (-15 + 48 + 0) - (6 - 0 + 18) = 33 - 24 = 9.$$

Properties of determinants

1. $\det A = \det A^T$;

2. Interchange of two rows (or columns) changes sign of determinant;

3. Common factor of all elements of row (or column) can be taken outside determinant;

4. Determinant with row (or column) of zeros is equal to 0;

5. Determinant with two equal rows (or columns) is equal to 0;

6. Determinant with two proportional rows (or columns) is equal to 0;

7. Determinant does not change if one row (or column) multiplied by constant is added to another row (or column).

Let us recalculate the determinant given in Example 2 by using properties.

Example 3 *Find* $\det A$ *if* $A = \begin{pmatrix} 5 & -2 & 1 \\ 3 & 1 & -4 \\ 6 & 0 & -3 \end{pmatrix}$.

Solution 3 *We use Property 7. Namely, the determinant does not change if we multiply the third column by* 2 *and add it to the first column. Then we apply "expansion by minors" with respect to the third row:*

$$\det A = \begin{vmatrix} 5 & -2 & 1 \\ 3 & 1 & -4 \\ 6 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 7 & -2 & 1 \\ -5 & 1 & -4 \\ 0 & 0 & -3 \end{vmatrix}$$

$$= (-1)^{3+3}(-3)\begin{vmatrix} 7 & -2 \\ -5 & 1 \end{vmatrix} = -3 \cdot (7 \cdot 1 - (-2) \cdot (-5)) = -3 \cdot (-3) = 9.$$

Inverse matrix

Definition 1 *The inverse of a square matrix* A *is a matrix* A^{-1} *such that*

$$AA^{-1} = A^{-1}A = I.$$

Theorem 1 *A square matrix A has an inverse if and only if* $\det A \neq 0$.

Suppose that $A = (a_{ij})_{n\times n}$. If $A^{-1} = (\alpha_{ij})_{n\times n}$, then

$$\alpha_{ij} = \frac{(-1)^{i+j} M_{ji}}{\det A}.$$

Example 4 *Find A^{-1} if* $A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$.

Solution 4 *First step is to find* $\det A$*:*

$$\det A = \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 2 \cdot 1 - 3 \cdot (-1) = 5.$$

Next, we find all elements of A^{-1}:

$$\alpha_{11} = \frac{(-1)^{1+1} M_{11}}{\det A} = \frac{1}{5};$$

$$\alpha_{12} = \frac{(-1)^{1+2} M_{21}}{\det A} = \frac{-3}{5};$$

$$\alpha_{21} = \frac{(-1)^{2+1} M_{12}}{\det A} = \frac{1}{5};$$

$$\alpha_{22} = \frac{(-1)^{2+2} M_{22}}{\det A} = \frac{2}{5}.$$

Thus,

$$A^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{-3}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}.$$

Example 5 *Find A^{-1} if* $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -5 & 3 \\ 2 & 7 & -1 \end{pmatrix}$.

Solution 5 *First step is to find* $\det A$*. If we multiply the first row by -3 and add to the second row, we get*

$$\det A = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -5 & 3 \\ 2 & 7 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -11 & 0 \\ 2 & 7 & -1 \end{vmatrix} = (-1)^{2+2}(-11)\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$$

$$= -11 \cdot (-1-2) = -11 \cdot (-3) = 33.$$

It is obvious that A^{-1} can be found as follows

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{pmatrix}^T .$$

Hence,

$$A^{-1} = \frac{1}{33} \begin{pmatrix} \begin{vmatrix} -5 & 3 \\ 7 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 3 & -5 \\ 2 & 7 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 7 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 7 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ -5 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} \end{pmatrix}^T$$

$$= \frac{1}{33} \begin{pmatrix} -16 & 9 & 31 \\ 9 & -3 & -3 \\ 11 & 0 & -11 \end{pmatrix}^T = \frac{1}{33} \begin{pmatrix} -16 & 9 & 11 \\ 9 & -3 & 0 \\ 31 & -3 & -11 \end{pmatrix} = \begin{pmatrix} -\frac{16}{33} & \frac{3}{11} & \frac{1}{3} \\ \frac{3}{11} & -\frac{1}{11} & 0 \\ \frac{31}{33} & -\frac{1}{11} & -\frac{1}{3} \end{pmatrix} .$$