Derivatives

Let us consider the graph of a function $y = f(x)$ that has a non-vertical tangent line at the point $M(x_0; y_0)$. Let α be an angle formed by this tangent line and the positive direction of the x-axis. Let us find the slope of the tangent line $k = \tan\alpha$. For this purpose, we draw one more line through two points of the graph $M(x_0; f(x_0))$ and $M_1(x_0 + \Delta x; f(x_0 + \Delta x))$ (Figure 1). This line MM_1 is a secant line of the graph. Let φ be an angle formed by this secant line and the positive direction of the x-axis. It is easy to find that

$$\tan\varphi = \frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

It is obvious that when Δx tends to 0, then Δy also tends to 0. It means that the point M_1 tends to the point M, therefore the secant line becomes closer to the tangent line. It means that φ becomes closer to α. This fact can be written as

$$\tan\alpha = \lim_{\Delta x \to 0} \tan\varphi.$$

Hence,

$$k = \tan\alpha = \lim_{\Delta x \to 0} \tan\varphi = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

This formula is the geometric interpretation of the derivative.

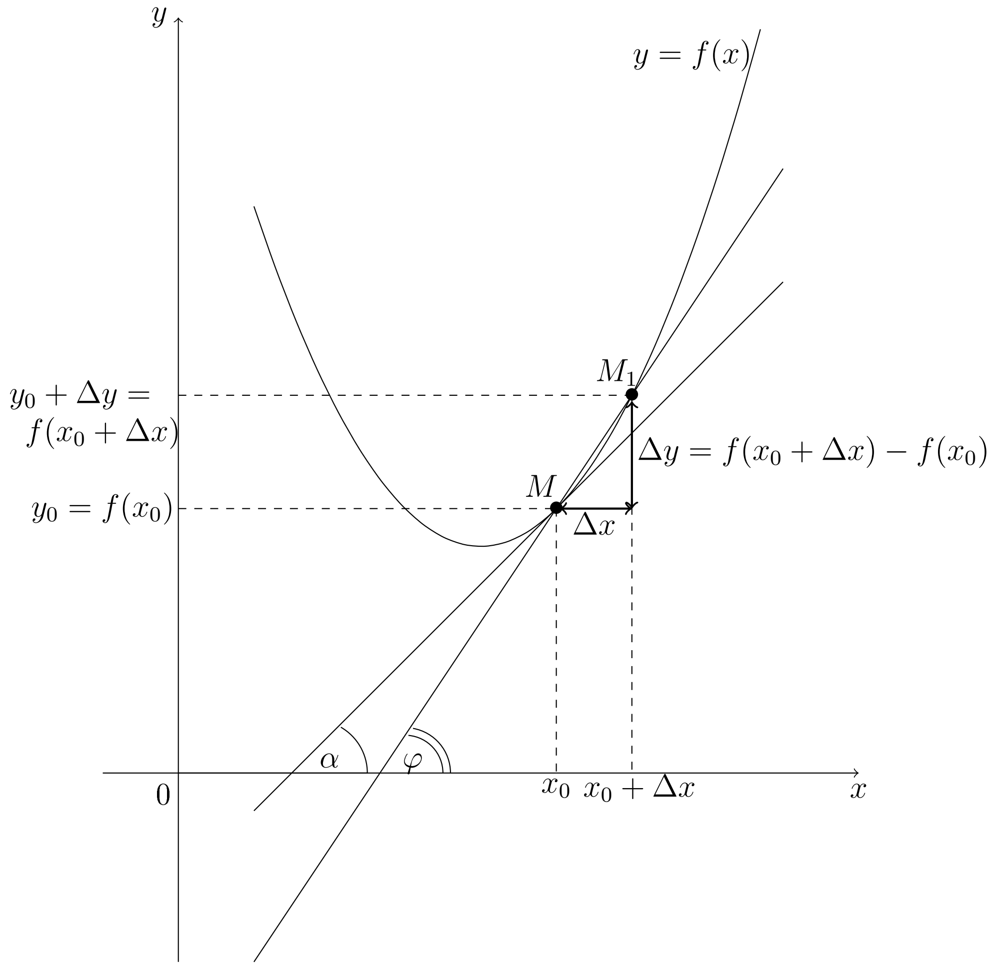


Figure 1

Definition 1 *The derivative of a function f is a new function f' defined by the formula*

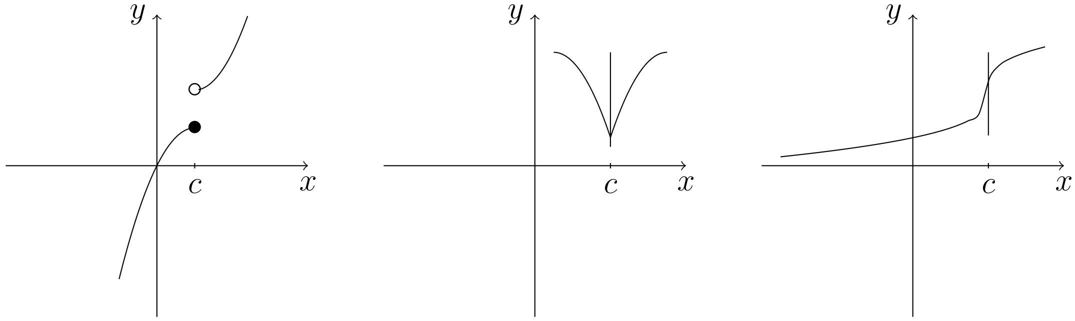
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

if the limit exists. If f' exists for each x in the interval $(a; b)$, then f is differentiable over $(a; b)$.

If the limit

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

does not exist, we say that f is non-differentiable at $x = c$. The points on the graph of f where $f'(c)$ does not exist can be recognized geometrically (Figure 2).



(A) not continuous at c (B) sharp corner at c (C) vertical tangent at c

Figure 2

Remark 1 *For a function $y = f(x)$, the notations*

$$f'(x), \quad y', \quad \frac{df}{dx}, \quad \frac{dy}{dx}, \quad Df(x) \quad \textit{and} \quad D_x y$$

all represent the derivative of f.

Example 1 *Find the derivative of $f(x) = x^2$.*

Solution 1 *By the definition of the derivative, we have*

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x.$$

Thus, $(x^2)' = 2x$.

Example 2 *Find the derivative of $f(x) = \ln x$.*

Solution 2 *By the definition of the derivative, we have*

$$f'(x) = \lim_{\Delta x \to 0} \frac{\ln(x+\Delta x) - \ln x}{\Delta x} = \lim_{\Delta x \to 0} \left(\frac{1}{\Delta x} \ln \frac{x + \Delta x}{x} \right)$$

$$= \lim_{\Delta x \to 0} \ln \left(\frac{x+\Delta x}{x} \right)^{\frac{1}{\Delta x}} = \lim_{\Delta x \to 0} \ln \left(1 + \frac{\Delta x}{x} \right)^{\frac{1}{\Delta x}}$$

$$= \lim_{\Delta x \to 0} \ln \left(\left(1 + \frac{\Delta x}{x} \right)^{\frac{x}{\Delta x}} \right)^{\frac{1}{x}} = \ln e^{\frac{1}{x}} = \frac{1}{x}.$$

Thus, $(\ln x)' = \frac{1}{x}$.

Arguing as in Examples 1 and 2 we construct

Table of main derivatives

1. $c' = 0$, where c is any constant;
2. $(x^\alpha)' = \alpha x^{\alpha-1}$;
3. $(a^x)' = a^x \ln a$, in particular, $(e^x)' = e^x$;
4. $(\log_a x)' = \frac{1}{x \ln a}$, in particular, $(\ln x)' = \frac{1}{x}$;
5. $(\sin x)' = \cos x$;
6. $(\cos x)' = -\sin x$;
7. $(\tan x)' = \frac{1}{\cos^2 x}$;
8. $(\cot x)' = -\frac{1}{\sin^2 x}$;
9. $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$;
10. $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$;
11. $(\arctan x)' = \frac{1}{1+x^2}$;
12. $(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$.

Chain rule

Let us develop a way to find the derivative of the composite function $y = f(u)$, where $u = g(x)$, i.e., $y = f(g(x))$.

For the function $y = f(u)$ we have that $\lim\limits_{\Delta u \to 0} \frac{\Delta y}{\Delta u} = y'_u$. Therefore, $\frac{\Delta y}{\Delta u} = y'_u + \alpha$ or

$$\Delta y = y'_u \cdot \Delta u + \alpha \cdot \Delta u,$$

where $\alpha \to 0$ when $\Delta u \to 0$.

Similarly, for the function $u = g(x)$ we have that $\lim\limits_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = u'_x$. Therefore,

$$\Delta u = u'_x \cdot \Delta x + \beta \cdot \Delta x,$$

where $\beta \to 0$ when $\Delta x \to 0$.

If we substitute the expression Δu in the expression Δy, we get

$$\Delta y = y'_u(u'_x \cdot \Delta x + \beta \cdot \Delta x) + \alpha(u'_x \cdot \Delta x + \beta \cdot \Delta x)$$

$$\Delta y = y'_u \cdot u'_x \cdot \Delta x + y'_u \cdot \beta \cdot \Delta x + \alpha \cdot u'_x \cdot \Delta x + \alpha \cdot \beta \cdot \Delta x.$$

Division by Δx yields

$$\frac{\Delta y}{\Delta x} = y'_u \cdot u'_x + y'_u \cdot \beta + \alpha \cdot u'_x + \alpha \cdot \beta,$$

that gives

$$y'_x = y'_u \cdot u'_x$$

when $\Delta x \to 0$.

This rule is known as the chain rule for the composite function $y = f(g(x))$, and it can be also written in the form:

$$y' = f'(g(x)) \cdot g'(x).$$

Rules of differentiation

Let $u(x)$ and $v(x)$ be differentiable functions over the interval $(a;b)$.

1. Sum – difference rule

Suppose that $y = u \pm v$. Then, by the definition of derivative, we have

$$\begin{aligned} y' &= \lim_{\Delta x \to 0} \frac{(u(x+\Delta x) \pm v(x+\Delta x)) - (u(x) \pm v(x))}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left(\frac{u(x+\Delta x) - u(x)}{\Delta x} \pm \frac{v(x+\Delta x) - v(x)}{\Delta x} \right) \\ &= \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \pm \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = u' \pm v'. \end{aligned}$$

Thus,

$$(u \pm v)' = u' \pm v'.$$

2. Product rule

Suppose that $y = u \cdot v$. Then

$$\begin{aligned} y' &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) \cdot v(x+\Delta x) - u(x) \cdot v(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(u(x)+\Delta u) \cdot (v(x)+\Delta v) - u(x) \cdot v(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x) \cdot v(x) + u(x) \cdot \Delta v + v(x) \cdot \Delta u + \Delta u \cdot \Delta v - u(x) \cdot v(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left(u(x) \cdot \frac{\Delta v}{\Delta x} + v(x) \cdot \frac{\Delta u}{\Delta x} + \Delta u \cdot \frac{\Delta v}{\Delta x} \right) \\ &= u(x) \cdot \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v(x) \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \Delta u \cdot \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \\ &= u \cdot v' + v \cdot u' + 0 \cdot v' = u' \cdot v + v' \cdot u. \end{aligned}$$

Thus,

$$(u \cdot v)' = u' \cdot v + v' \cdot u.$$

In particular,

$$(c \cdot u)' = c \cdot u',$$

where c is a constant.

3. Quotient rule

Suppose that $y = \frac{u}{v}$. Then

$$y' = \lim_{\Delta x \to 0} \frac{\frac{u(x+\Delta x)}{v(x+\Delta x)} - \frac{u(x)}{v(x)}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{u(x)+\Delta u}{v(x)+\Delta v} - \frac{u(x)}{v(x)}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x) \cdot v(x) + v(x) \cdot \Delta u - u(x) \cdot v(x) - u(x) \cdot \Delta v}{\Delta x (v(x) + \Delta v) v(x)}$$

$$= \lim_{\Delta x \to 0} \frac{v(x) \cdot \Delta u - u(x) \cdot \Delta v}{\Delta x (v^2(x) + v(x) \cdot \Delta v)} = \lim_{\Delta x \to 0} \frac{v(x) \cdot \frac{\Delta u}{\Delta x} - u(x) \cdot \frac{\Delta v}{\Delta x}}{v^2(x) + v(x) \cdot \Delta v}$$

$$= \frac{v(x) \cdot \lim\limits_{\Delta x \to 0} \frac{\Delta u}{\Delta x} - u(x) \cdot \lim\limits_{\Delta x \to 0} \frac{\Delta v}{\Delta x}}{v^2(x) + v(x) \cdot \lim\limits_{\Delta x \to 0} \Delta v} = \frac{u' \cdot v - v' \cdot u}{v^2}.$$

Thus,

$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - v' \cdot u}{v^2}.$$

Example 3 *Find the derivative of* $f(x) = 5^{x^3 - 4}$.

Solution 3 *We use the chain and difference rules:*

$$f'(x) = (5^{x^3-4})' = 5^{x^3-4} \cdot \ln 5 \cdot (x^3 - 4)' = 5^{x^3-4} \cdot \ln 5 \cdot 3x$$

Example 4 *Find the derivative of* $f(x) = \frac{3x^2+4x-5}{\cos x}$.

Solution 4 *We use the quotient and sum-difference rules:*

$$f'(x) = \left(\frac{3x^2 + 4x - 5}{\cos x}\right)' = \frac{(3x^2 + 4x - 5)' \cos x - (\cos x)'(3x^2 + 4x - 5)}{\cos^2 x}$$

$$= \frac{(6x + 4) \cos x + \sin x (3x^2 + 4x - 5)}{\cos^2 x}$$

Higher order derivatives

If a function $y = f(x)$ has the derivative f', then the derivative of f', if it exists, is called the second order derivative and written as f''. The derivative of f'', if it exists, is called the third order derivative and written as f''', and so on. If we continue this process we can find nth order derivative.

Remark 2 *For a function $y = f(x)$, the notations*

$$f''(x), \quad y'', \quad \frac{d^2 f}{dx^2}, \quad \frac{d^2 y}{dx^2}, \quad D_x^2 f(x) \ \ \text{and} \ \ D_x^2 y$$

all represent the second order derivative of f.

The third order derivative is written similarly. For $n \geq 4$, the nth order derivative is written as $f^{(n)}(x)$.

Example 5 *Find the fourth order derivative of $f(x) = 7x^4 + 3x^3 + 5x^2 - 6x + 11$.*

Solution 5

$$f'(x) = 28x^3 + 9x^2 + 10x - 6;$$

$$f''(x) = 84x^2 + 18x + 10;$$

$$f'''(x) = 168x + 18;$$

$$f^{(4)}(x) = 168.$$