Functions

In mathematics, the central concept of function is correspondence. Thus, a function is some relation between a set of inputs and a set of outputs. Usually, to denote that y is a function of x we write $y = f(x)$, where x is called the *independent* variable and y is called the *dependent* variable.

As an example of a function, let us consider the following well-known formula:

$$s = \pi r^2.$$

This formula is the relationship between the radius r and the area of the circle s. It is obvious that the larger r the larger s. Therefore, it is a function that can be presented in the form $s = f(r)$.

The second function example is as follows:

$$t = \begin{cases} 1852, & 0 < s \leq 1100, \\ 3704, & 1100 < s \leq 1500, \\ 5556, & 1500 < s \leq 2000, \\ 11112, & 2000 < s \leq 2500, \\ 16668, & 2500 < s \leq 3000, \\ 27780, & 3000 < s \leq 4000, \\ 216684, & s > 4000. \end{cases}$$

This formula shows the correspondence between the tax rate t on vehicles (in tenge) and their engine size s in Kazakhstan in 2014. This function can be written in the form $t = f(s)$.

Definition 1 *A function f is a rule that assigns to each element (number) x of a set $D(f)$ a unique element (number) of a set $R(f)$.*

The set $D(f)$ is called the domain of f and the set $R(f)$ is called the range of f.

Example 1 *Find the domains of the functions:*

1. $f(x) = \sqrt{1 - x^2}$; *2.* $g(x) = \frac{1}{x-3}$.

Solution 1 *1. For the square root to exist,* $1-x^2$ *must be greater than or equal to* 0*. That is,*

$$1-x^2 \geq 0 \ \ or \ \ (1-x)(1+x) \geq 0$$

$$\begin{cases} 1-x \geq 0 \\ 1+x \geq 0 \end{cases} or \ \begin{cases} 1-x \leq 0 \\ 1+x \leq 0 \end{cases}$$

$$\begin{cases} x \leq 1 \\ x \geq -1 \end{cases} or \ \begin{cases} x \geq 1 \\ x \leq -1 \end{cases}.$$

The second system has no solutions. Thus, $D(f) = [-1;1]$*.*

2. For the fraction to exist, the denominator $x-3$ *must be not equal to* 0 *(division by* 0 *is not defined). That is,*

$$x-3 \neq 0 \ \ or \ \ x \neq 3.$$

Thus, $D(f) = (-\infty;3) \cup (3;+\infty)$*.*

Cartesian system of coordinates

The set of all pairs of all real numbers is called the number plane. This number plane can be represented by *Cartesian system of coordinates* (Figure 1).

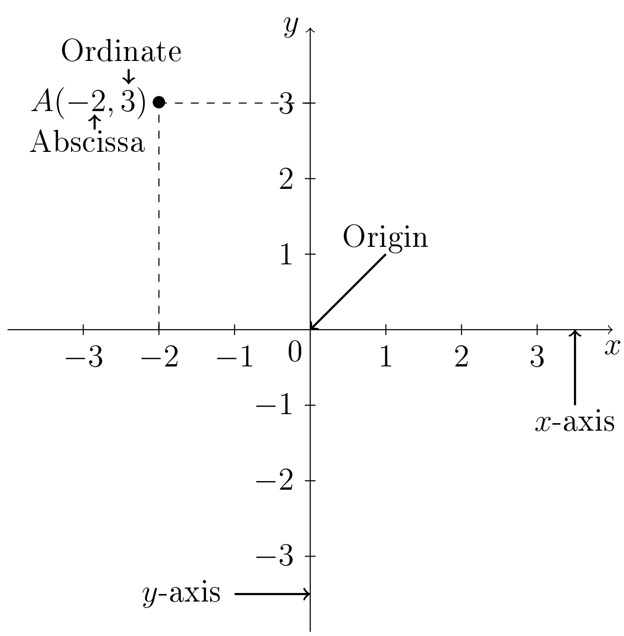


Figure 1

The graph of a function f consists of those points in the Cartesian plane whose coordinates (x, y) satisfy the equation $y = f(x)$. It means that the pair (x, y) lies on the graph of f if and only if x is of the domain and $y = f(x)$. Usually, to draw the graph of a function we use a table of coordinate pairs $(x, f(x))$ for various values of x in the domain of f, then plot these points and connect them with a "smooth" curve.

Example 2 *Sketch the graph of the function* $f(x) = (x-1)^3 + 2$.

Solution 2 *Make a table of coordinate pairs that satisfy the equation* $f(x) = (x-1)^3 + 2$*:*

x	*-1*	*0*	*1*	*2*	*3*
y	*-6*	*1*	*2*	*3*	*10*

Plot these points and join them with a smooth curve as shown in Figure 2.

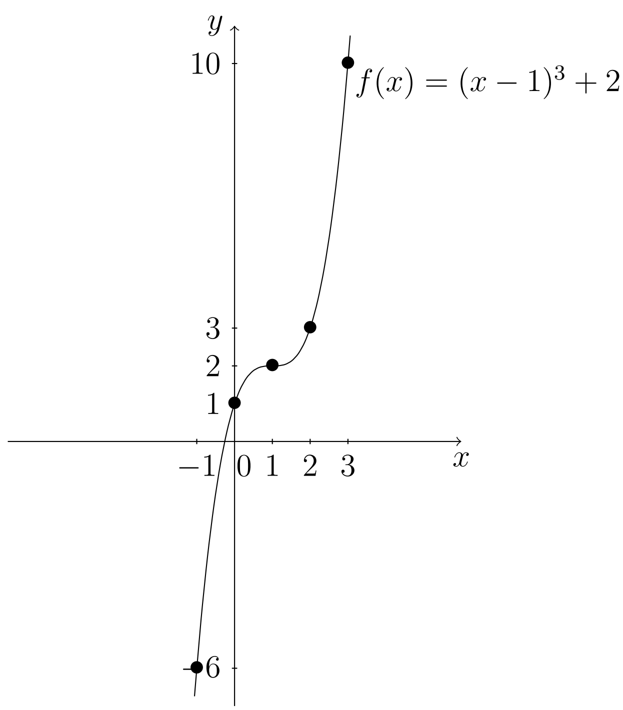


Figure 2

Graph transformations

Comparing the graph of $y = f(x) + k$ with the graph of $y = f(x)$, we see that it is the graph of $y = f(x)$ vertically shifted up by k units if k is positive and down by k units if k is negative.

Comparing the graph of $y = f(x+h)$ with the graph of $y = f(x)$, we see that it is the graph of $y = f(x)$ horizontally shifted left by h units if h is positive and right by h units if h is negative.

To sketch the graph of $y = -f(x)$ we reflect the graph of $y = f(x)$ in the x-axis.

Moreover, the graph of $y = Af(x)$ is a vertical stretch of the graph of $y = f(x)$ if $A > 1$ and a vertical shrink of the graph of $y = f(x)$ if $0 < A < 1$.

Remark 1 *The graph of the equation* $f(x) = (x-1)^3 + 2$ *given in Example 2 can be sketched by using graph transformations. Thus, first we sketch the graph of* $f(x) = x^3$*. Then we shift it upward by* 2 *units and to the right by* 1 *unit.*

Intercepts

x-intercept (root) is the abscissa of a such point of the graph, where this graph crosses (touches) the x-axis. To find x-intercept we pose $y = 0$, and x satisfying $f(x) = 0$ will be x-intercept.
y-intercept is the ordinate of a such point of the graph, where this graph crosses (touches) the y-axis. To find y-intercept we pose $x = 0$, and y satisfying $y = f(0)$ will be y-intercept.

Increasing and decreasing functions

We say that a function f is *increasing* on an interval $(a;b)$ if $f(x_1) < f(x_2)$ whenever $a < x_1 < x_2 < b$. We say that a function f is *decreasing* on an interval $(a;b)$ if $f(x_1) > f(x_2)$ whenever $a < x_1 < x_2 < b$.

Some elementary functions

Linear function

Definition 2 *A function f is a linear function if*

$$y = kx + b,$$

where k and b are real numbers.

The domain and range of a linear function is the set of all real numbers.

The graph of a linear function $y = kx + b$ is a straight line with the *slope* k and y*-intercept* b. Therefore, the equation $y = kx + b$ is called the *slope-intercept form* of a linear function.

The *standard form* of a linear function is

$$Ax + By = C,$$

where A, B and C are real numbers. If $B \neq 0$, the standard equation can be resolved with respect to y:

$$y = -\frac{A}{B}x + \frac{C}{B}$$

that is the slope-intercept form.

Suppose that two fixed distinct points $M_1(x_1; y_1)$ and $M_2(x_2; y_2)$ belong to the graph of $y = kx + b$. It means that their coordinates satisfy the equation $y = kx + b$; hence, $y_1 = kx_1 + b$ and $y_2 = kx_2 + b$. If we find the difference $y_2 - y_1$, we get

$$y_2 - y_1 = (kx_2 + b) - (kx_1 + b) = kx_2 + b - kx_1 - b = k(x_2 - x_1)$$

$$y_2 - y_1 = k(x_2 - x_1)$$

$$k = \frac{y_2 - y_1}{x_2 - x_1}.$$

Definition 3 *If a straight line passes through two distinct points $M_1(x_1; y_1)$ and $M_2(x_2; y_2)$, then its slope is given by the formula:*

$$k = \frac{y_2 - y_1}{x_2 - x_1}.$$

Thus, the slope k of a line $y = kx + b$ equals to the tangent of an angle α formed by this line and the positive direction of the x-axis. It is known that

(A) if $0 < \alpha < \frac{\pi}{2}$ (α is an acute angle), then $\tan\alpha > 0$;

(B) if $\frac{\pi}{2} < \alpha < \pi$ (α is an obtuse angle), then $\tan\alpha < 0$;

(C) if $\alpha = 0$, then $\tan\alpha = 0$;

(D) if $\alpha = \frac{\pi}{2}$, then $\tan\alpha$ does not exist.

Therefore, since $k = \tan\alpha$, we have

(A) if $k > 0$, then a straight line $y = kx + b$ is increasing;

(B) if $k < 0$, then a straight line $y = kx + b$ is decreasing;

(C) if $k = 0$, then a straight line $y = b$ is a horizontal line parallel to the x-axis;

(D) if k does not exist, then $x = a$ is a vertical line that is not a function.

Each case is illustrated in Figure 3.

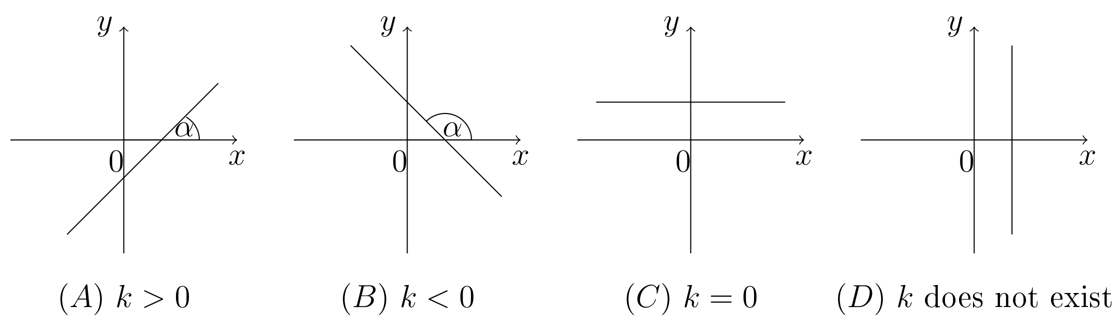


Figure 3

Arguing as above, for a line that passes through an arbitrary point $M(x; y)$ and a fixed point $M_1(x_1; y_1)$, its slope is given by the formula:

$$k = \frac{y - y_1}{x - x_1}.$$

Thus, we write the following definition.

Definition 4 *If a straight line passes through the point $M_1(x_1; y_1)$ and has the slope k, then its equation is given by the formula:*

$$y - y_1 = k(x - x_1).$$

Moreover, if we combine

$$k = \frac{y_2 - y_1}{x_2 - x_1} \text{ and } k = \frac{y - y_1}{x - x_1},$$

we come up with one more definition.

Definition 5 *If a straight line passes through two distinct points* $M_1(x_1; y_1)$ *and* $M_2(x_2; y_2)$*, then its equation is given by the formula:*

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

Parallel and perpendicular lines

Suppose that we have two distinct lines $y_1 = k_1 x + b_1$ and $y_2 = k_2 x + b_2$. Denote by α and β the angles formed by the lines y_1 and y_2 and positive direction of the x-axis, respectively. It means that $k_1 = \tan\alpha$ and $k_2 = \tan\beta$.

1. If two non-vertical lines are parallel, they have the same slope.

Since α and β are two corresponding angles formed by two parallel lines y_1 and y_2 and the x-axis as a transversal line (Figure 4), then $\alpha = \beta$. Therefore, $k_1 = k_2$.

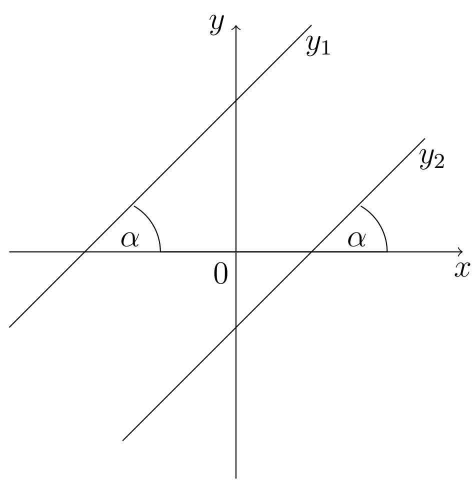


Figure 4

2. If two lines y_1 and y_2 are perpendicular, excluding the case of vertical and horizontal lines, the product of their slopes is equal to -1.

To prove this formula we draw a line parallel to the x-axis through the point of intersection of y_1 and y_2 (Figure 5). Since the sum of two interior angles formed by two parallel lines and a transversal line equals to π, we have

$$\left(\frac{\pi}{2} - \alpha\right) + \beta = \pi$$

$$\beta = \frac{\pi}{2} + \alpha.$$

We substitute the obtained relation in k_2 and get

$$k_2 = \tan\beta = \tan(\frac{\pi}{2} + \alpha) = -\cot\alpha = -\frac{1}{\tan\alpha} = -\frac{1}{k_1}.$$

Therefore, $k_1 \cdot k_2 = -1$.

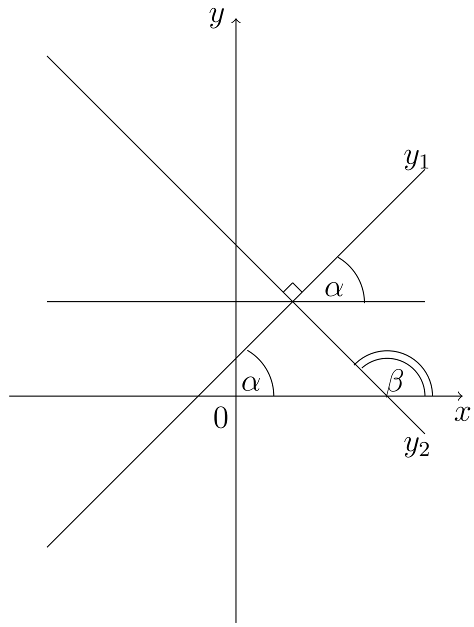


Figure 5

Example 3 *Find the equation of the line*

1. if it passes through the points $(1; -2)$ *and* $(-4; 3)$*;*

2. if it passes through the point $(5; -3)$ *and is parallel to the line* $3x - 2y = 6$*;*

3. if it is perpendicular to the line $y = \frac{1}{3}x + \frac{4}{7}$ *and has the y-intercept (5).*

Solution 3 *1. If we use Definition 5, we get*

$$\frac{y+2}{3+2} = \frac{x-1}{-4-1}$$

$$(y+2)\cdot(-5)=(x-1)\cdot 5$$

$$y+2=-x+1$$

$$y=-x-1.$$

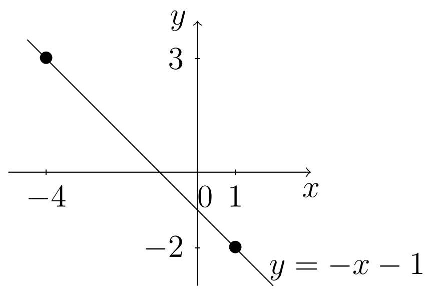


Figure 6

2. If we resolve the equation $3x-2y=6$ with respect to y, we get $y=\frac{3}{2}x-3$. The slope of this line is $\frac{3}{2}$, it means that the slope of the desired function is also $\frac{3}{2}$. Now, we use Definition 4 and obtain

$$y+3=\frac{3}{2}(x-5)$$

$$y=\frac{3}{2}x-\frac{15}{2}-3$$

$$y=\frac{3}{2}x-\frac{21}{2}.$$

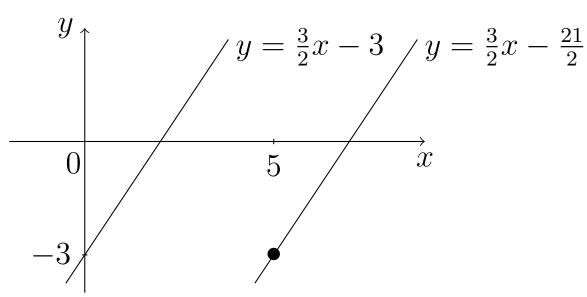


Figure 7

3. The slope of the perpendicular line is $\frac{1}{3}$. Since the product of slopes of two perpendicular lines is (-1), the slope of the desired line is (-3). By the condition, the y-intercept is (5). Thus, the desired equation is

$$y=-3x+5.$$

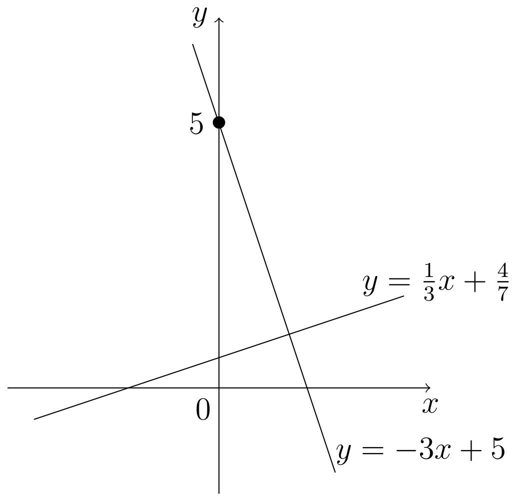


Figure 8

Applications

The supply and demand of an item are usually related to its price. Producer will supply larger numbers of the item at a higher price. However, at a higher price, consumer will demand less of the item. Thus, in general, the graph of the supply equation $y = S(x)$ increases and the graph of the demand equation $y = D(x)$ decreases. Here, x stands for the quantity and y stands for the price. If we have linear functions for supply and demand curves, they can be written in the form:

$$S(x) = ax + b$$

and

$$D(x) = mx + n,$$

where a, b, m and n are real numbers. Moreover, since the line $S(x)$ increases and the line $D(x)$ decreases, their slopes must be positive ($a > 0$) and negative ($m < 0$), respectively. The price tends to stabilize at the point of intersection of the supply and the demand equations. This is *the equilibrium point*, where its first coordinate x is *the equilibrium quantity* and its second coordinate y is *the equilibrium price.*

Example 4 *At a price of \$50 per kilo, the annual Kazakhstan supply and demand for tea are 1500 and 1700 tones, respectively. When the price rises to \$80, the supply* *increases to 1800 tones while the demand decreases to 1300 tones. 1. Assuming that the price-supply and the price-demand equations are linear, find their equations; 2. Find the equilibrium point for Kazakhstan tea market.*

Solution 4 *1. We have that the supply curve $S(x)$ passes through two points $(1500; 50)$ and $(1800; 80)$. If we use Definition 5, we get*

$$\frac{S(x) - 50}{80 - 50} = \frac{x - 1500}{1800 - 1500}$$

$$(S(x) - 50) \cdot 300 = (x - 1500) \cdot 30 \quad or \quad (S(x) - 50) \cdot 10 = x - 1500$$

$$S(x) = \frac{1}{10}x - 100.$$

Similarly, since the demand curve $D(x)$ passes through two points $(1700; 50)$ and $(1300; 80)$, from Definition 5 we have

$$\frac{D(x) - 50}{80 - 50} = \frac{x - 1700}{1300 - 1700}$$

$$(D(x) - 50) \cdot (-400) = (x - 1700) \cdot 30 \quad or \quad (D(x) - 50) \cdot 40 = (x - 1700) \cdot (-3)$$

$$40 \cdot D(x) = -3x + 7100$$

$$D(x) = -\frac{3}{40}x + \frac{355}{2}.$$

2. To find the equilibrium point, we need to solve the equation $S(x) = D(x)$. Thus,

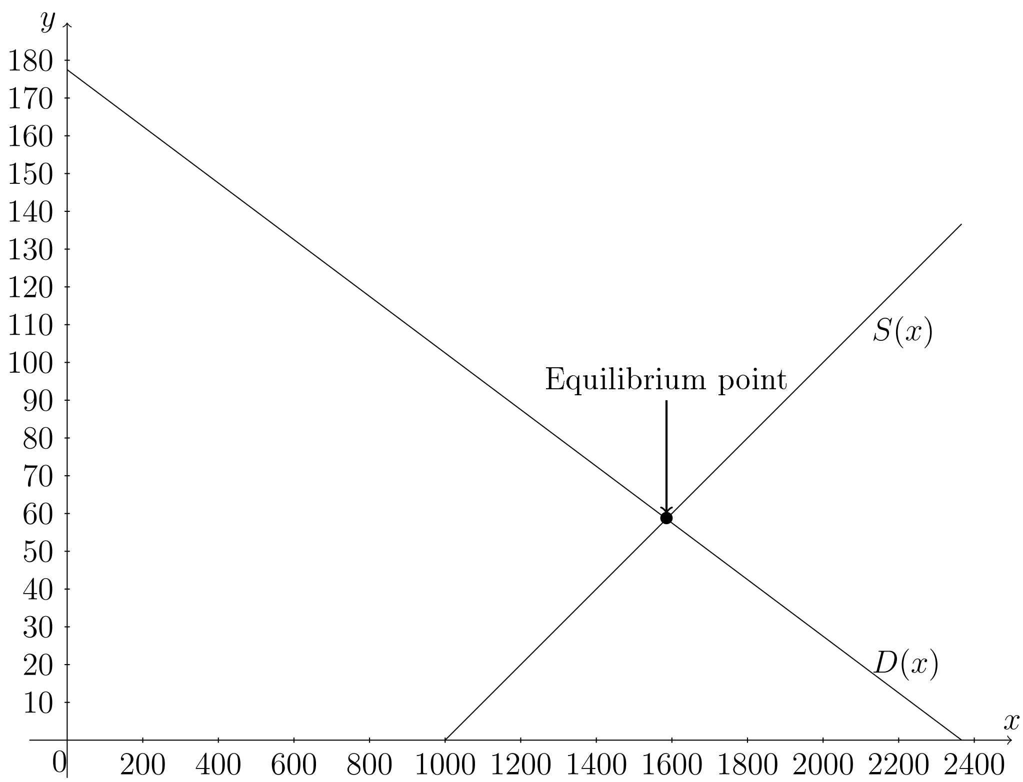
$$\frac{1}{10}x - 100 = -\frac{3}{40}x + \frac{355}{2}$$

$$\frac{1}{10}x + \frac{3}{40}x = \frac{355}{2} + 100 \quad or \quad \frac{7}{40}x = \frac{555}{2}$$

$$x = \frac{555}{2} \cdot \frac{40}{7} = \frac{11100}{7} \approx 1585.7 \;\; tones.$$

Moreover,

$$S\left(\frac{11100}{7}\right) = D\left(\frac{11100}{7}\right) = \frac{1}{10} \cdot \frac{11100}{7} - 100 = \frac{1110}{7} - \frac{700}{7} = \frac{410}{7} \approx \$58.57 \;\; per\ kilo.$$



y
180
170
160
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
0
200
400
600
800
1000
1200
1400
1600
1800
2000
2200
2400
x
S(x)
D(x)
Equilibrium point

Figure 9