Problems for the constrained optimization (Chapter #12)

Example 19.7. In example 18.5, we considered the problem of maximizing $f(x_1,x_2)=x_1^2x_2$ on the constraint set $h(x_1,x_2)=2x_1^2+x_2^2=3.$ There we found six solutions to the first conditions

$$(x_1,x_2,\mu)=\begin{cases}(0,\pm\sqrt{3},0)\\(\pm 1,+1,+.5)\\(\pm 1,-1,-.5)\end{cases}$$

Let us use second order conditions to decide which of these points are local maxima and which are local minima.

Differentiate the first order conditions once again to obtain the general bordered Hessian

$$H=\begin{pmatrix}0 & h_{x_1} & h_{x_2}\\ h_{x_1} & L_{x_1x_1} & L_{x_1x_2}\\ h_{x_1} & L_{x_2x_1} & L_{x_2x_2}\end{pmatrix}=\begin{pmatrix}0 & 4x_1 & 2x_2\\ 4x_1 & 2x_2-4\mu & 2x_1\\ 2x_2 & 2x_1 & -2\mu\end{pmatrix}$$

This problem has n = 2 variables and $k=1$ equality constraint. As Theorem 19.7 indicates, we need only check the sign as $(-1)^n = +1$, that is, if det H>0, at a candidate point, that point is a local max. If det H has the same sign as $(-1)^n = -1$, that is, if det H<0, at a candidate point, that point is a local min.

At the points $(\pm 1,-1,-0.5)$,

$$H=\begin{pmatrix}0 & \pm 4 & 2\\ \pm 4 & 0 & \pm 2\\ -2 & \pm 2 & 1\end{pmatrix}$$

In either case, det H = -16; so these two points are local minima.

At the points $(\pm 1,1,0.5)$,

$$H=\begin{pmatrix}0 & \pm 4 & 2\\ \pm 4 & 0 & \pm 2\\ 2 & \pm 2 & -1\end{pmatrix}$$

In either case, det H = +48; so these two points are local maxima.

This computations support the observations we made at the end of example 18.5. However, we were not able to determine the character of $(x_1,x_2)=(0,\pm\sqrt{3})$ by simply plugging these points into the objective function in Example 18.5. Since $\mu = 0$ for these points, the corresponding bordered Hessian is

$$H = \begin{pmatrix} 0 & 0 & \pm 2\sqrt{3} \\ 0 & \pm 2\sqrt{3} & 0 \\ \pm 2\sqrt{3} & 0 & 0 \end{pmatrix}$$

For $(x_1, x_2) = (0, +\sqrt{3})$, det H = $-24\sqrt{3} < 0$; this point is a local min. For $(x_1, x_2) = (0, -\sqrt{3})$, det $H =$ $+24\sqrt{3} > 0$; this point is a local max.

Example 19.8 max $x^2 y^2 z^2$
subject to $x^2 + y^2 + z^2 = 3$

$$\frac{\partial L}{\partial x} = 2xy^2z^2 - 2\mu x = 0$$
$$\frac{\partial L}{\partial y} = 2x^2yz^2 - 2\mu y = 0$$
$$\frac{\partial L}{\partial z} = 2x^2y^2z - 2\mu z = 0$$
$$\frac{\partial L}{\partial x} = x^2 + y^2 + z^2 - 3 = 0$$

with solution $x^2 = y^2 = z^2 = \mu = 1$. The bordered Hessian:

$$\begin{pmatrix} 0 & 2x & 2y & 2z \\ 2x & 2y^2z^2 - 2\mu & 4xyz^2 & 4xy^2z \\ 2y & 4xyz^2 & 2x^2z^2 - 2\mu & 4x^2yz \\ 2z & 4xy^2z & 4x^2yz & 2y^2x^2 - 2\mu \end{pmatrix}$$

At $x = y = z = \mu = 1$, the bordered Hessian becomes

$$\left(\begin{array}{ccc|c} 0 & 2 & 2 & 2 \\ 2 & 0 & 4 & 4 \\ 2 & 4 & 0 & 4 \\ \hline 2 & 4 & 4 & 0 \end{array}\right)$$

Since n = 3 and k = 1, we have to check the signs of two leading principal minors: the matrix 3×3 submatrix H_3 above the dashed lines and the complete 4×4 matrix H_4. One computes that H_3 = 32 and det H_4 = -192. Since these determinants Alternate in sign and since the sign of det H_4 is the sign of $(-1)^3$ = - 1, the candidate $x = y = z = 1$ is indeed local constrained max by Theorem19.6.

Problem 14-1 (14.1)

Find the largest value $U = x^2 y$ can attain when $3x + 4y = 72$, $x \geq 0$, $y \geq 0$, and find the corresponding values of x and y.

Problem 14-2 (14.1)

Solve the utility maximization problem max $100x^{1/2}y^{1/4}$ subject to $px + qy = m$

Problem 14-3 (14.2)

Consider the utility maximization problem

$$\max\ 2x_1x_2 + 3x_1 \quad \text{subject to} \quad x_1 + 2x_2 = 83$$

(a) Write down the Lagrangian and the first-order conditions. Find the only possible solution to the problem.

(b) What is the approximate change in the optimal value function if 83 is changed to 84?

Problems from Chapter 10:

Exercise 10.7 (1) 10.6 (1); 10.4 (4,5); 10.2(4)

Problems for Chapter 11:

Exercise 11.6(4)

Problem 13-1 (13.2)

A firm produces $Q = K^{1/2}L^{1/4}$ units of its output good when it uses K units capital and L units of labor. The firm obtains the price 16 per unit of output and pays 4 per unit of capital and 2 per unit of labor.

(a) Is Q homogeneous in K and L?

(b) Find the firm's profit.

(c) Find the values of K and L that maximize profit.

Problem 13-2 (13.2)

Let g be a function of two variables defined for all x and y by

$$g(x, y) = 20x^2 + 12xy + 2y^2 + 4x + 2y - 1$$

(a) Find $g(0, 2)$, $g(-2, 3)$, and $g(a + h, 1) - g(a, 1)$.

(b) Find the first- and second-order partials of g.

(c) Find the minimum point of g.

Problem 13-3 (13.1)

A function of two variables is given by

$$f(x, y) = 5x^2 - 2xy + 2y^2 - 4x - 10y + 5 \qquad \text{for all } x \text{ and } y$$

Find the partial derivatives of f of the first and second order. Find the stationary points.

A firm produces two commodities, A and B. The inverse demand functions are

$$p_A = 900 - 2x - 2y \qquad \text{and} \qquad p_B = 1400 - 2x - 4y$$

respectively, where the firm produces and sells x units of commodity A and y units of commodity B. The costs are given by

$$C_A = 7\,000 + 100x + x^2 \qquad \text{and} \qquad K_B = 10\,000 + 6y^2$$

(a) Show that the firm's profit is given by

$$\pi(x, y) = -3x^2 - 10y^2 - 4xy + 800x + 1\,400y - 17\,000$$

(b) Find the values of x and y that maximize profits.

(c) Suppose the firm is required to produce a total of exactly 60 units. Find the values of x and y that maximize profits.