Lecture Notes for Chapter 9

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1 Optimization of One Variable

1.1 Critical Points

A critical point occurs whenever the firest derivative of a function is equal to zero.

ie.

$$\begin{aligned} \text{If } y &= f(x) \\ \text{Then } \frac{dy}{dx} &= f'(x) = 0 \text{ is a critical point} \end{aligned}$$

A critical point is a "stationary" value of the function (flat spot).

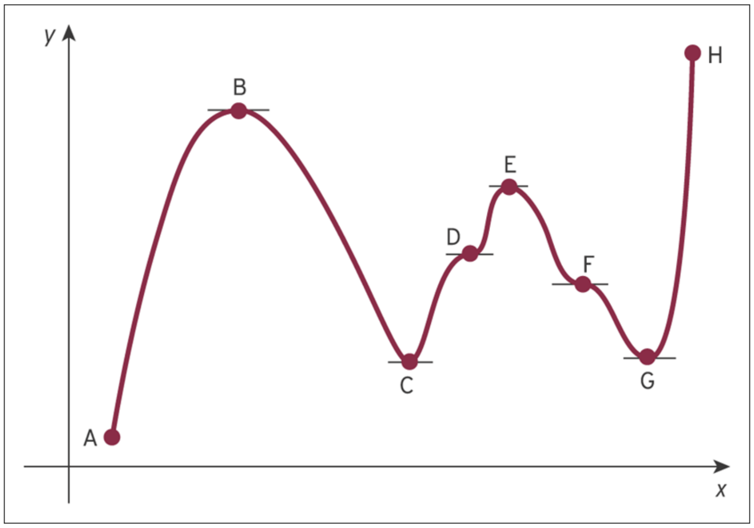
A critical pointcan be:

a) some maximum point

b) some minimum point

c) an inflection point, which is neitehr a max or a min.

1.2 Relative versus Global Points



1.3 The First Derivative Test for a Relative Extremum

$f'(x) = 0$ is a necessary, but not a sufficient condition to establish an extremum.

Test: if, at $x = x_0$, $f'(x) = 0$, then the value of $f(x)$ will be:

1. (a) Relative Maximum

$$\begin{aligned} \text{if } f'(x) &> 0 \text{ for } x < x_0 \\ \text{and } f'(x) &< 0 \text{ for } x > x_0 \end{aligned}$$

   (b) Relative Minimum

$$\begin{aligned} \text{if } f'(x) &< 0 \text{ for } x < x_0 \\ \text{and } f'(x) &> 0 \text{ for } x > x_0 \end{aligned}$$

   (c) Inflection Point

$$\text{if } f'(x) \text{ has the same sign for } x > x_0 \text{ and } x < x_0$$

Example: Total Revenue Let: $TR = 10q - q^2$
Then: $MR = \frac{dTR}{dq} = 10 - 2q$ where $MR = 0$ at $q = 5$

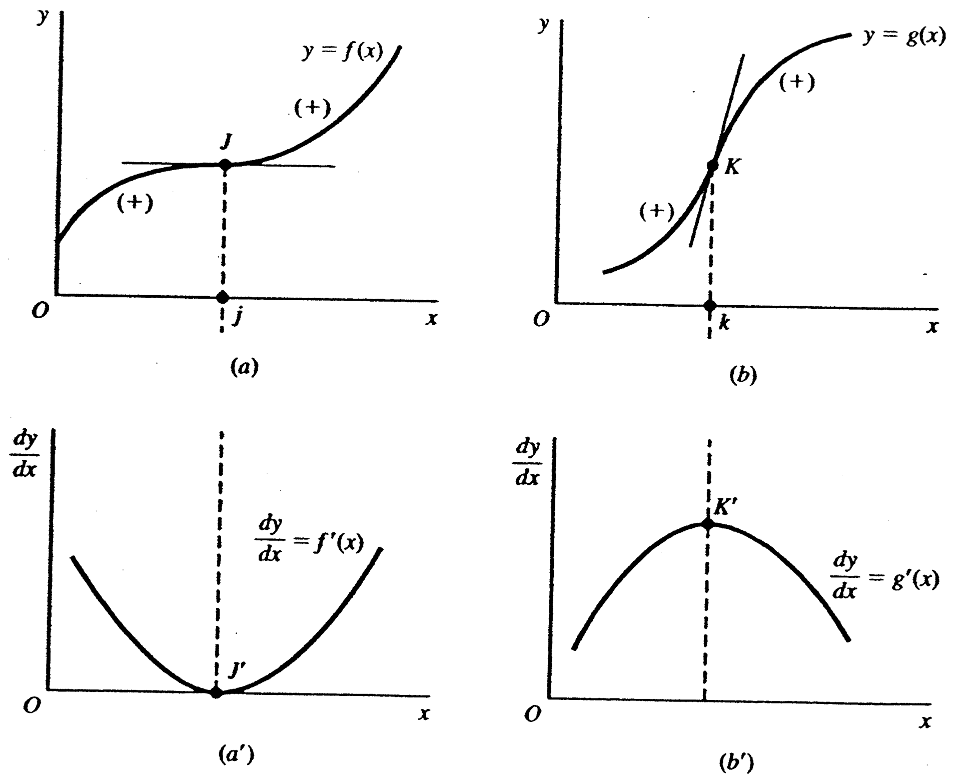
$$\begin{aligned} Let &: TR = 10q - q^2 \\ Then &: MR = \frac{dTR}{dq} = 10 - 2q \text{ where } MR = 0 \text{ at } q = 5 \\ AR &= \frac{TR}{q} \\ AR &= \frac{10q - q^2}{q} \\ AR &= 10 - q \end{aligned}$$

[1]

1.4 More on Inflection Points

Inflection points do not occur only when $f'(x) = 0$

[1] Insert graph (x2) on page 4 Chapter 9



Example: Production Function [2]

[2] Insert graph page 6 (x 2)

1.5 Second and Higher Derivatives

$$\begin{aligned}
\text{Given } y &= f(x) \text{ is a function of x} \\
\text{Then } \frac{dy}{dx} &= f'(x) \text{ is also a function of x} \\
\text{We can find the derivative of } f'(x) &: \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d^2y}{dx^2} = f''(x) \\
\text{is the second derivative of } y &= f(x)
\end{aligned}$$

Similarly:

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = f'''(x) \text{ or } f^{(3)}(x)$$

is the third derivative and $f^{(4)}$ is the fourth....and $f^{(n)}(x)$ is the nth derivative.

1.6 Interpretation of the Second Derivative

For a function

$$y = f(x)$$

$f'(x)$ measures the rate of change of the function
$f''(x)$ measures the rate of change of the rate of change of the function
Evaluated at $x = x_0$

$f'(x_0) > 0$ means that the value of the function increases
$f'(x_0) < 0$ means that the value of the function decreases
Whereas:
$f''(x_0) > 0$ means that the slope of the curve increases
$f''(x_0) < 0$ means that the slope of the curve decreases

Example: $t = \text{time}$

$$\begin{aligned} \text{if Distance} &= f(t) \\ \text{then Velocity} &= f'(t) \\ \text{and Acceleration} &= f''(t) \end{aligned}$$

1.6.1 Concavity and Convexity

fig (a) Slope of the function always decreasing ($f'' < 0$)
fig (b)Slope always increasing ($f'' > 0$)

1.7 Second Derivative Test

Test:

$$\begin{aligned} \text{Given} \quad y &= f(x) \quad \text{and} \quad x = x_0 \\ \text{If} \quad f'(x_0) &= 0 \end{aligned}$$

Then: $f(x_0)$ will be:

1. (a) a relative maximum if $f''(x_0) < 0$
   (b) a relative minimum if $f''(x_0) > 0$
   (c) either a maximum, or a minimum, or an inflection point if $f''(x_0) = 0$

If (c) then use either the first derivative test or some other test
In economics, the second derivative test is sufficient 99% of the time (the other 1% appears on finals)

$$EV = P \cdot \pi_1 + (1 - P)\,\pi_2$$

[3]

[3]not sure how to do the decision tree type of thing on page 13

Example: P=1/2 1-P=1/2 $\pi_1 = 9$ $\pi_2 = 25$
$EV = (1/2)(9) + (1/2)(25) = 17$
$EU = W^{1/2}$

1.7.1 Profit Maximization

Suppose a certain producer is a monopolist

$$\begin{aligned} \text{Let} &: R = R(Q) \text{ be his total revenue function} \\ \text{and let} &: C = C(Q) \text{ be his total cost function} \\ \text{Therefore, profits}\,(\pi) \text{ are} &: \pi(Q) = R(Q) - C(Q) \end{aligned}$$

Necessary condition for profit maximization

$$\frac{d\pi}{dQ} = R'(Q) - C'(Q) = 0$$

or

$$\begin{aligned} MR &= MC \\ & \text{(Min or Max)} \end{aligned}$$

Sufficient condition for profit maximization

$$\begin{aligned} \frac{d^2\pi}{dQ^2} &= R''(Q) - C''(Q) < 0 \\ \text{or } R'' &< C'' \end{aligned}$$

[4]

$$\begin{aligned} \pi &= \pi(Q) = R(Q) - C(Q) \\ \frac{d\pi}{dQ} &= \pi' = R'(Q) - C'(Q) = 0 \\ \text{or } R'(Q) &= C'(Q) \qquad \text{MR=MC} \end{aligned}$$

[4] Insert 3 graphs on page 15

2nd derivative test

$$\begin{aligned} \frac{d^2\pi}{dQ^2} &= R''(Q) - C''(Q) < 0 \\ &\text{or } R''(Q) < C''(Q) \end{aligned}$$

[5]

Example 1

$$\begin{aligned} R(Q) &= 1200Q - 2Q^2 \\ C(Q) &= Q^3 - 61.25Q^2 + 1528.5Q + 2000 \\ \pi(Q) &= R(Q) - C(Q) \\ &= -Q^3 + 59.25Q^2 - 328.5Q - 2000 \\ \frac{d\pi}{dQ} &= -3Q^2 + 118.5Q - 328.5 = 0 \end{aligned}$$

2 Solutions: $Q = 3,\ 36.5$
2nd Derivative Test

$$\frac{d\pi^2}{dQ^2} = -6Q + 118 \qquad \begin{cases} > 0 & Q = 3 \\ < 0 & Q = 36.5 \end{cases}$$

Example of Concave π from $R - C$
[6]

1.8 Nth Derivative Test

Suppose at $x = x_{0,}$ $f'(x_0) = 0$ and at $x = x_0$, $f''(x_0) = 0$ What then?

[5] insert graphs on page 16
[6] Insert graph page 17

Test: Keep differentiating until you come to the first NONZERO derivative. This derivative will be the Nth derivative of $f(x_0)$. Then $f(x_0)$ will be

1. (a) a relative max if N is even and $f^N(x_0) < 0$
   (b) a relative min if N is even and $f^N(x_0) > 0$
   (c) an inflection point if N is odd

Example Evaluate $y = (7-x)^4$ at $x = 7$

$$\begin{aligned}
&\Rightarrow \; f'(x) = -4(7-x)^3 \quad \Rightarrow f'(7) = 0 \\
&\Rightarrow \; f''(x) = 12(7-x)^2 \quad \Rightarrow f''(7) = 0 \\
&\Rightarrow \; f^3(x) = -24(7-x) \quad \Rightarrow f^3(7) = 0 \\
&\Rightarrow \; f^4(x) = 24 \quad \Rightarrow f^4(7) = 24
\end{aligned}$$

Since $N = 4$ is even, and $f^4 = 24 > 0$, then $f(7)$ is a <u>relative minimum</u>.

Next Week: Read MacLaurin and Taylor series (sect. 9.5) and chapter 11.

1.9 Taylor Series

Consider the function

$$f(x) = a + bx + cx^2 + dx^3 + ex^4$$

Lets evaluate f and its derivatives at the point $x = 0$

$$\begin{aligned}
f(x) &= a + bx + cx^2 + dx^3 + ex^4 & f(0) = a \\
f'(x) &= b + 2cx + 3dx^2 + 4ex^3 & f'(0) = b \\
f''(x) &= 2c + 6dx + 12ex^2 & f''(0) = 2c \\
f^{(3)}(x) &= 6d + 24ex & f^{(3)}(0) = 6d \\
f^{(4)}(x) &= 24e & f^{(4)}(0) = 24e \\
\text{OR} \quad a &= f(0) \quad b = f'(0) \quad c = \frac{f''(0)}{2} \quad d = \frac{f^{(3)}(0)}{6} \quad e = \frac{f^4(0)}{24}
\end{aligned}$$

Now sub into $f(x)$ the values of a, b, c, d, e

$$\begin{aligned}
f(x) &= a + bx + cx^2 + dx^3 + ex^4 \\
f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 + \frac{f^4(0)}{24}x^4
\end{aligned}$$

Factorials:

$$\begin{aligned}
n! &= nx(n-1)x(n-2)x...x(3)x(2)x1 \\
2! &= 2x1 = 2 \\
3! &= 3x2x1 = 6 \\
4! &= 4x3x2x1 = 24 \\
&\quad \text{etc.}
\end{aligned}$$

Therefore:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4$$

Taylor Series Expansion

1. In the neighbourhood of $x = x_0$ we can approximate some function $g(x)$ with a polynomial function, $f(x)$

2. At point $x = x_0$ our approximation should have certain properties that are common to $g(x)$:

   Specifically at:

$$\begin{aligned} x &= x_0 \\ f(x_0) &= g(x_0) \\ f'(x_0) &= g'(x_0) \\ f''(x_0) &= g''(x_0) \\ f^3(x_0) &= g^3(x_0) \\ &\dots \\ f^n(x_0) &= g^n(x_0) \end{aligned}$$

3. Even though we do not know the exact form of $g(x)$ if we know the properties of $g(x)$ at $x = x_0$ then we can approximate $g(x)$ with a polynomial.

1.9.1 Polynomial "approximation" of g(x) around x=x$_0$

$$\begin{aligned} g(x_0) &\approx f(x) = a + bx + cx^2 + dx^3 + \dots \\ f(x) &= (g\overset{a}{(x_0)}) + (g'\overset{b}{(x_0)})(x - x_0) + \left(\frac{g''\overset{c}{(x_0)}}{2!}\right)(x - x_0)^2 \\ &\quad + \left(\frac{g^3\overset{d}{(x_0)}}{3!}\right)(x - x_0)^3 + \dots \end{aligned}$$

if $x = x_0$

then

$$f\left(x_0\right) = g\left(x_0\right) + g'\left(x_0\right)(0) + \left(\frac{g''\left(x_0\right)}{2!}\right)(0) + ...$$

treat g, g', g'', g^3 etc. as constants!

$$f(x) = g'\left(x_0\right) + \frac{2g''\left(x_0\right)}{2!}(x - x_0) + \frac{3g^3\left(x_0\right)}{3!}(x - x_0)^2 + ...$$

at $x = x_0$

$$\begin{aligned} f'(x) &= g'(x_0) \text{ and the other terms drop out} \\ f''(x) &= g''\left(x_0\right) + \frac{6g^3\left(x_0\right)}{3!}(x - x_0) + ... \end{aligned}$$

at $x = x_0$

$$f''(x) = g''\left(x_0\right) \text{ and the other terms drop out}$$

Read the section explaining "The Remainder" of a Taylor Series