**Lecture 8**

**Differentials and Derivatives**

$^{dy}/\_{dx}=f^{'}\left(x\right)=\lim\_{∆x\to y}\frac{∆y}{∆x}$ (by definition)

$\frac{∆y}{∆x}-\frac{dy}{dx}=δ$ where $δ\rightarrow 0$, $∆x\rightarrow 0$

$∆y=\frac{dy}{dx}∆x+δ∆x$ or $∆y=f^{'}\left(x\right)\*∆x+δ∆x$

this equation describes the change in y ($∆y)$ that results from a specific not necessarily small change in x ($∆x)$ from any starting value of x in the domain of the function y.

 y=f(x)

y B

 $δ∆x$

 D

 $∆y$ $f^{'}(x\_{0})∆x$

 A C

 $∆x$

 x

 x0

Y y=f(x)

 dy

 A dx

 x

$\frac{CB}{AC}=\frac{∆y}{∆x} $ AD – a tangent line

DB - the discrepancy of error of approximation as $∆x$ decreases B →A making $f^{'}\left(x\right) or \frac{dy}{dx}$ a better approximation of $\frac{∆y}{∆x}$.

 $\frac{dy}{dx}$ - slope of tangent AD=f'(x)

$$dy=f^{'}\left(x\right)dy$$

**Differentials and Point Elasticity**

$Q=f(P)$ - demand function

$E\_{p}=^{\frac{∆Q}{Q}}/\_{\frac{∆P}{P}}$ $∆Q\rightarrow dQ$, $∆P\rightarrow dP$

$$E\_{p}=^{\frac{∆Q}{Q}}/\_{\frac{∆P}{P}}=\frac{^{dQ}/\_{dP}\rightarrow derivative}{^{Q}/\_{P}\rightarrow average fun.}$$

-ratio of the marginal function to the average function of the demand function.

For other function y=f(x) the point elasticity of y with respect to x.

$$E\_{xy}=\frac{\frac{dy}{dx}}{\frac{y}{x}}=\frac{marginal function}{average function}$$

Example 1.

$$Q=100-2P$$

$^{dQ}/\_{dP}=-2$ $^{Q}/\_{P}=\frac{100-2P}{P}$

$$E\_{p}\left(D\right)=^{-2}/\_{\frac{(100-2P)}{P}=\frac{-2P}{100-2P}=\frac{-P}{50-P}}$$

$E\_{p}(P)$= -1

$$\frac{-P}{50-P}=-1$$

$$-P=-50+P=>-2P=-50,P=25$$

$\left|E\_{p}\right|$>1 =>25<p<50

$\left|E\_{p}\right|$>1 =>0<p<25

Example 2.

Find the point elasticity of supply $E\_{p}\left(S\right);$ $Q=P^{2}+7P$, whether the supply is elastic at P=2

$^{dQ}/\_{dP}=2P+7$ $^{Q}/\_{P}=P+7$ $E\_{p}\left(S\right)=\frac{2P+7}{P+7 }$

P=2, $E\_{p}\left(S\right)=\frac{2\*2+7}{2+7 }=\frac{11}{9}>1$=> supply is elastic

y

 A

 QA QM

0 X0 B x **Inelastic**

$$Q\_{M}<Q\_{A}$$

y y=f(x)

$ Q\_{M}>Q\_{A}$

 A

 QA QM x  **Elastic**

 **0**

**y**

 C

 QA QM

0 X0 x QA=QM

**Unitary Elastic**

If AB is steeper the OA, the function is elastic at point A; in the opposite case it is inelastic at A.

Find the differentials for given functions:

1) a) $y=-x(x^{2}+3)$

$$y^{'}=-\left(x^{2}+3\right)-x\left(2x\right)=-3x^{2}-3=-3(x^{2}+1)$$

$dy=-3(x^{2}+1)dx$

b) $y=\left(x-8\right)\left(7x+5\right)$

$$y^{'}=7x+5+\left(x-8\right)7=14x-51$$

$$dy=(14x-51)dx$$

c) $y=\frac{x}{x^{2}+1}$

2)$M=f(Y)$ => $E\_{y}\left(M\right)=^{\frac{∆M}{∆Y}}/\_{\frac{M}{Y}}=\frac{^{dM}/\_{dY}}{^{M}/\_{Y}}$

3) C=a+bY

$$\frac{C}{Y}=\frac{a}{Y}+b$$

$\frac{dC}{dY}$=b

$$E\_{y}\left(C\right)=\frac{b}{\frac{a+bY}{Y}}=\frac{Yb}{a+bY};$$

for $Y>0$; a>0; 0<a<1; b>0 => Yb<a+bY=>$E\_{y}\left(C\right)<1$

4) $Q=100-2P+0.02Y$

Q- quantity demanded, P- price, Y -income,

given P=20, Y=5000 find the:

a) price elasticity of demand

b) income elasticity of demand

$$Q'\_{p}=-2$$

$$\frac{Q}{P}=\frac{100-2P+0.02Y}{P}$$

$$\frac{Q}{Y}=\frac{100-2P+0.02Y}{Y}$$

$E\_{p}\left(D\right)=\frac{-2P}{100-2P+0.02Y}=\frac{-2\*20}{100-2\*20+0.02\*5000}=\frac{-40}{100-40+100}=\frac{-40}{160}=\left|-0.25\right|<1$ price inelastic.

$E\_{y}\left(D\right)=\frac{0.02\*Y}{100-2P+0.02Y}=\frac{100}{160}=\frac{5}{8}=0.625>0$ Income elasticity is positive, so the product is normal good.

**Total Differentials**

The concept of differentials can easily be extended to a function of 2 or more independent variables. S=s(Y,i) - saving function - to this function is assumed to be continuous and to possess continuous ( partial) derivatives.

$ds=\frac{∂s}{∂y}dy+\frac{∂s}{∂i}di$ -Total differential of the saving function.

The process of finding such a total differential is called a total differentiation.

It is possible, that Y may change while i remains constant. In that case, di=0, $ds=\left(^{∂s}/\_{∂y}\right)dy\~$

$$\frac{dS}{dy}=\left(\frac{∂s}{∂y}\right)\_{i=const}$$

More general case

U=U($x\_{1},x\_{2},…,x\_{n}) $ - if this utility function $\frac{∂U}{∂x\_{i}}$ , marginal utility, $\frac{∂U}{∂x\_{i}}dx\_{i}, i=1,n-is the approximate change in U usually from a change in x\_{i}.$

$$dU=\frac{∂U}{∂x\_{1}}dx\_{1}+\frac{∂U}{∂x\_{2}}dx\_{2}+…+\frac{∂U}{∂x\_{n}}dx\_{n}$$

$$dU=U\_{1}dx\_{1}+U\_{2}dx\_{2}+…+U\_{n}dx\_{n}=\sum\_{i=1}^{n}U\_{i}dx\_{i}$$

$$E\_{sy}=\frac{^{∂s}/\_{∂y}}{^{s}/\_{y}}=\frac{∂s}{∂y}\*\frac{y}{s}$$

$$E\_{si}=\frac{\frac{∂s}{∂i}}{\frac{s}{i}}=\frac{∂s}{∂i}\*\frac{i}{s}$$

$$E\left(U\right)\_{x\_{i}}=\frac{∂U}{∂x\_{i}}\*\frac{x\_{i}}{U}$$

These are partial elasticity.

Example 1. Find the tottal differential for the following utility functions where a,b>0

a)$U\left(x\_{1},x\_{2}\right)=ax\_{1}+bx\_{2}$

b)$U\left(x\_{1},x\_{2}\right)=x\_{1}^{2}+x\_{2}^{3}+x\_{1}x\_{2}$

c)$ U\left(x\_{1},x\_{2}\right)=x\_{1}^{a}x\_{2}^{b}$

**Total derivatives**

y=f(x,w) where x=g(w)

y=f[g(w),w]

$$\frac{dy}{dw}=f\_{x}\frac{dx}{dw}+f\_{w}=\frac{∂y}{∂x}\frac{dx}{dw}+\frac{∂y}{∂w}$$

$\frac{dy}{dw}$ - total derivative, $\frac{∂y}{∂w}$ - partial derivative

Example 1. Find the total derivative dy/dw

$$y=f\left(x,w\right)=3x-w^{2}$$

$$x=2w^{2}+w+4$$

$$x=g\left(w\right)$$

$$\frac{dy}{dw}=3\left(4w+1\right)+\left(-2w\right)=10w+3$$

As a check, we may substitute the function g into the function f, to get $y=3\left(2w^{2}+w+4\right)-w^{2}=5w^{2}+3w+12$

$$\frac{dy}{w}=10w+3$$

A variation

$y=f\left(x\_{1},x\_{2},w\right)$ where $\left\{\begin{array}{c}x\_{1}=g\left(w\right)\\x\_{2}=h\left(w\right)\end{array}\right.$

$$\frac{dy}{dw}=\frac{∂y}{∂x\_{1}}\frac{dx\_{1}}{dw}+\frac{∂y}{∂x\_{2}}\frac{dx\_{2}}{dw}+\frac{∂y}{∂w}=f\_{1}\frac{dx\_{1}}{dw}+f\_{2}\frac{dx\_{2}}{dw}+f\_{w};$$

Another variation

$y=f\left(x\_{1},x\_{2},u,v\right)$ where $x\_{1}=y\left(u,v\right)$

$$x\_{2}=g(u,v)$$

$$\frac{dy}{du}=\frac{∂y}{∂x\_{1}}\frac{dx\_{1}}{du}+\frac{∂y}{∂x\_{2}}\frac{dx\_{2}}{du}+\frac{∂y}{∂u}\frac{du}{du}+\frac{∂y}{∂v}\frac{dv}{du}=\frac{∂y}{∂x\_{1}}\frac{dx\_{1}}{du}+\frac{∂y}{∂x\_{2}}\frac{dx\_{2}}{dy}+\frac{∂y}{∂u}$$

$$\left[\frac{dv}{du}=0, v-hold const.\right]$$

Example. $Q=Q(K,L,t)$, let be the production function, where, aside from the two inputs K and L, there is a third argument t, denoting time. This is a dynamic production function,

K=K(t),

L=L(t)

$$\frac{dQ}{dt}=\frac{∂Q}{∂K}\frac{dK}{dt}+\frac{∂Q}{∂L}\frac{dL}{dt}+\frac{∂Q}{∂t}$$