SOLUTIONS QUESTION 1

Problems and Applications

1. a. A production function has constant returns to scale if increasing all factors of production by an equal percentage causes output to increase by the same percentage. Mathematically, a production function has constant returns to scale if $zY = F(zK, zL)$ for any positive number z. That is, if we multiply both the amount of capital and the amount of labor by some amount z, then the amount of output is multiplied by z. For example, if we double the amounts of capital and labor we use (setting $z = 2$), then output also doubles.

   To see if the production function $Y = F(K, L) = K^{1/2}L^{1/2}$ has constant returns to scale, we write:

$$F(zK, zL) = (zK)^{1/2}(zL)^{1/2} = zK^{1/2}L^{1/2} = zY.$$

   Therefore, the production function $Y = K^{1/2}L^{1/2}$ has constant returns to scale.

   b. To find the per-worker production function, divide the production function $Y = K^{1/2}L^{1/2}$ by L:

$$\frac{Y}{L} = \frac{K^{1/2}L^{1/2}}{L}.$$

   If we define $y = Y/L$, we can rewrite the above expression as:

$$y = K^{1/2}/L^{1/2}.$$

   Defining $k = K/L$, we can rewrite the above expression as:

$$y = k^{1/2}.$$

c. We know the following facts about countries A and B:

δ = depreciation rate = 0.05,
s_a = saving rate of country A = 0.1,
s_b = saving rate of country B = 0.2, and
$y = k^{1/2}$ is the per-worker production function derived in part (b) for countries A and B.

The growth of the capital stock Δk equals the amount of investment $sf(k)$, less the amount of depreciation δk. That is, $\Delta k = sf(k) - \delta k$. In steady state, the capital stock does not grow, so we can write this as $sf(k) = \delta k$.

To find the steady-state level of capital per worker, plug the per-worker production function into the steady-state investment condition, and solve for k^*:

$$sk^{1/2} = \delta k.$$

Rewriting this:

$$k^{1/2} = s/\delta$$
$$k = (s/\delta)^2.$$

To find the steady-state level of capital per worker k^*, plug the saving rate for each country into the above formula:

$$\text{Country A:} \quad k_a^* = (s_a/\delta)^2 = (0.1/0.05)^2 = 4.$$

$$\text{Country B:} \quad k_b^* = (s_b/\delta)^2 = (0.2/0.05)^2 = 16.$$

Now that we have found k^* for each country, we can calculate the steady-state levels of income per worker for countries A and B because we know that $y = k^{1/2}$:

$$y_a^* = (4)^{1/2} = 2.$$
$$y_b^* = (16)^{1/2} = 4.$$

We know that out of each dollar of income, workers save a fraction s and consume a fraction $(1 - s)$. That is, the consumption function is $c = (1 - s)y$. Since we know the steady-state levels of income in the two countries, we find

$$\text{Country A:} \quad c_a^* = (1 - s_a)y_a^* = (1 - 0.1)(2) = 1.8.$$

$$\text{Country B:} \quad c_b^* = (1 - s_b)y_b^* = (1 - 0.2)(4) = 3.2.$$

d. Using the following facts and equations, we calculate income per worker y, consumption per worker c, and capital per worker k:

$s_a = 0.1.$
$s_b = 0.2.$
$\delta = 0.05.$
$k_o = 2$ for both countries.
$y = k^{1/2}.$
$c = (1 - s)y.$

Country A

Year	k	$y = k^{1/2}$	$c = (1 - s_a)y$	$i = s_a y$	δk	$\Delta k = i - \delta k$
1	2	1.414	1.273	0.141	0.100	0.041
2	2.041	1.429	1.286	0.143	0.102	0.041
3	2.082	1.443	1.299	0.144	0.104	0.040
4	2.122	1.457	1.311	0.146	0.106	0.040
5	2.102	1.470	1.323	0.147	0.108	0.039

Country B

Year	k	$y = k^{1/2}$	$c = (1 - s_a)y$	$i = s_a y$	δk	$\Delta k = i - \delta k$
1	2	1.414	1.131	0.283	0.100	0.183
2	2.183	1.477	1.182	0.295	0.109	0.186
3	2.369	1.539	1.231	0.308	0.118	0.190
4	2.559	1.600	1.280	0.320	0.128	0.192
5	2.751	1.659	1.327	0.332	0.138	0.194

Note that it will take five years before consumption in country B is higher than consumption in country A.

SOLUTIONS QUESTION 2

3. a. We follow Section 7-1, "Approaching the Steady State: A Numerical Example." The production function is $Y = K^{0.3}L^{0.7}$. To derive the per-worker production function $f(k)$, divide both sides of the production function by the labor force L:

$$\frac{Y}{L} = \frac{K^{0.3}L^{0.7}}{L}.$$

Rearrange to obtain:

$$\frac{Y}{L} = \left(\frac{K}{L}\right)^{0.3}.$$

Because $y = Y/L$ and $k = K/L$, this becomes:

$$y = k^{0.3}.$$

b. Recall that

$$\Delta k = sf(k) - \delta k.$$

The steady-state value of capital k^* is defined as the value of k at which capital stock is constant, so $\Delta k = 0$. It follows that in steady state

$$0 = sf(k) - \delta k,$$

or, equivalently,

$$\frac{k^*}{f(k^*)} = \frac{s}{\delta}.$$

For the production function in this problem, it follows that:

$$\frac{k^*}{(k^*)^{0.3}} = \frac{s}{\delta}.$$

Rearranging:

$$(k^*)^{0.7} = \frac{s}{\delta},$$

or

$$k^* = \left(\frac{s}{\delta}\right)^{1/0.7}$$

Substituting this equation for steady-state capital per worker into the per-worker production function from part (a) gives:

$$y^* = \left(\frac{s}{\delta}\right)^{0.3/0.7}$$

Consumption is the amount of output that is not invested. Since investment in the steady state equals δk^*, it follows that

$$c^* = f(k^*) - \delta k^* = \left(\frac{s}{\delta}\right)^{0.3/0.7} - \delta\left(\frac{s}{\delta}\right)^{1/0.7}$$

(*Note:* An alternative approach to the problem is to note that consumption also equals the amount of output that is not saved:

$$c^* = (1-s)f(k^*) = (1-s)(k^*)^{0.3} = (1-s)\left(\frac{s}{\delta}\right)^{0.3/0.7}$$

Some algebraic manipulation shows that this equation is equal to the equation above.)

c. The table below shows k^*, y^*, and c^* for the saving rate in the left column, using the equations from part (b). We assume a depreciation rate of 10 percent (i.e., 0.1). (The last column shows the marginal product of capital, derived in part (d) below).

	k^*	y^*	c^*	*MPK*
0	0.00	0.00	0.00	
0.1	1.00	1.00	0.90	0.30
0.2	2.69	1.35	1.08	0.15
0.3	4.80	1.60	1.12	0.10
0.4	7.25	1.81	1.09	0.08
0.5	9.97	1.99	1.00	0.06
0.6	12.93	2.16	0.86	0.05
0.7	16.12	2.30	0.69	0.04
0.8	19.50	2.44	0.49	0.04
0.9	23.08	2.56	0.26	0.03
1	26.83	2.68	0.00	0.03

Note that a saving rate of 100 percent (s = 1.0) maximizes output per worker. In that case, of course, nothing is ever consumed, so c^* =0. Consumption per worker is maximized at a rate of saving of 0.3 percent—that is, where s equals capital's share in output. This is the Golden Rule level of s.

d. We can differentiate the production function $Y = K^{0.3}L^{0.7}$ with respect to K to find the marginal product of capital. This gives:

$$MPK = 0.3\frac{K^{0.3}L^{0.7}}{K} = 0.3\frac{Y}{K} = 0.3\frac{y}{k}.$$

The table in part (c) shows the marginal product of capital for each value of the saving rate. (Note that the appendix to Chapter 3 derived the *MPK* for the general Cobb–Douglas production function. The equation above corresponds to the special case where α equals 0.3.)