**Lecture 5**

**Differentiation**

**The Derivative**

 The derivative is a fundamental calculus concept and is a useful tool in economics. To carefully understand the derivative, the concept of a limit and the concept of a tangent must first be understood.

 Suppose the variable *y* depends upon the variable *x* as described by the function *f(•)* so that *y = f(x)*. The *limit* of the function *f(•)* evaluated at *x = x0*, written is the value approached by the variable *y* as an independent variable *x* approaches the value *x0*.

 If *point* is on the graph of a function *f(•)*, then the line *tangent* to the graph of *f(•)* at point *P* is defined to be the line through *P* having the slope



See Figure A.2 for an illustration of a line tangent to the graph of a function *f(•)* at point .

 The *derivative* of the function *f(x)* is the function *f’(x)* defined by



That is, the derivative of the function *f(•)* evaluated at *x* is equal to the slope of the line tangent to the graph of *f(•)* at the point *[x,f(x)]*. Geometrically, the derivative *f’(x)* is the slope of the curve *f(x).* The geometry of the derivative’s definition is shown in Figure A.2. In the figure, the derivative is being defined for the point .

**Figure A.2: Defining the Derivative Geometrically**



























 Intuitively, the derivative gives the change in the dependent variable that occurs when the independent variable increases by one unit, (where one unit is infinitesimally small.) The following derivative rules are useful in obtaining derivatives. The rules are given under the assumption that *y* is a dependent variable, *x* is an independent variable, and *c* is a constant.

*Constant Rule:* If *y = c* then

.

For example, if *y = 5,* then

.

*Power Rule*: *y = xc* then

.

For example, if *y = x5* , then



A special case in which the power rule is used is when *y = x*. In

this case,



*Constant Coefficient Rule*: If *y = cf(x),* then



For example, if *y = 5x* then



*Sum Rule*: If *y = f(x) + g(x)* then



For example, if *y = 5x + 2x2* , then



*Product Rule*: If *y = f(x)g(x),* then



For example, if *y = [5x][2x],* then



*Natural Log Rule*: If *y =*ln(*x*), then



For example, if *y=5*[ln(*x*)], then



*Chain Rule*: If *y = f(g(x)),* then



For example, if *y =* (*3+4x*)*2*  then



 In economics, a given dependent variable typically depends upon more than just one independent variable. For example, rather than a variable *y* merely depending upon a single variable *x*, the variable *y* may depend upon three variables *x*1, *x*2, and *x*3 so that *y = f(x1,x2,x3*.*)*. To find out how a change in one of the independent variables affects the dependent variable, partial differentiation can be used. The *partial derivative* ∂*y*/∂*x*1 gives the change in the dependent variable y that occurs when the independent variable *x*1 increases by one unit, assuming that the other independent variables *x*2 and *x*3 are constant.

Knowing which variables are to be treated as constant and which are allowed to vary, partial derivatives can be calculated using the rules of differentiation described above. For example, suppose that



To find out how the variable *x*1 affects *y*, take the derivative treating *x*1 as a variable, *x*2 as a constant, and *x*3 as a constant:



 , Sum Rule

 , Constant Rule

 , Constant Coefficient Rule

  Power Rule

To find out how the variable *x*2 affects *y*, take the derivative treating *x*2 as a variable, *x*1 as a constant, and *x*3 as a constant:



 , Sum Rule

 , Constant Rule

 , Constant Coefficient Rule

  Power Rule

Finally, to find out how the variable *x*3 affects *y*, take the derivative treating *x*3 as a variable, *x*1 as a constant, and *x*2 as a constant:



 , Sum Rule

 , Constant Rule

 , Constant Coefficient Rule

  Power Rule

**Implicit Differentiation**

 Consider the equation. Here, we can think of  as a function of , or we can think of  as a function of . If we choose the latter, we could solve for  and obtain the *explicit function* . That is, , where  is the function. The derivative  can then be found as .

 An alternative way of finding this derivative is to use *implicit differentiation.* Implicit differentiation. Implicit differentiation is based upon the notion of the *implicit function.* When we write , we are emphasizing the we are thinking of  as depending upon (i.e.,  is a function of ). However, when we write, we can still say  is a function of . When we say this, we are recognizing a functional relationship that is implicit, but not explicit. To implicitly differentiate the condition , we differentiate each side of the equation first, and then solve for .

 To illustrate, we implicitly differentiate the equation  by first writing

.

We have applied the product rule of differentiation here because in the condition,  is obviously a function of, and we are assuming  is a function of . These two functions of  are in a multiplicative relationship, so we must apply the product rule. The next step is to apply the differentiation rules and obtain.

.

Finally, we need to solve for . Doing so, we obtain

.

Because, we can eliminate  from this last expression and obtain, which is the same result we obtained from directly differentiating the explicit function.

**Differentials**

 The symbols and  in the derivative  are called differentials. The symbol  can be interpreted as the change in the variable  that occurs as the variable  changes by . Because the derivative is the slope of the line tangent to the function at the point in question, these changes approximate the movement along the function . This approximation is perfect when the function is linear. When the function is nonlinear, the change in  (i.e., ) becomes less accurate as the change in  (i.e., ) becomes larger.

 The point here is that we can find derivates we are interest in by first taking differentials. The rules for finding differentials are analogous to those for finding derivatives.

*Constant Rule:* If *y = c* then

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For example, if *y = 5,* then

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*Power Rule*: *y = xc* then

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*Constant Coefficient Rule*: If *y = cf(x),* then



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*Sum Rule*: If *y = f(x) + g(x)* then



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*Product Rule*: If *y = f(x)g(x),* then



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*Natural Log Rule*: If *y =*ln(*x*), then



For example, if *y=5*[ln(*x*)], then



**Total Differentiation**

 Conceptually, total differentiation involves letting all variables in an equation change. Reconsider the equation . If  changes, then  must change in order for the equation to still hold. If and  each change, then we must use the product rule. The differentiation would be as follows.





.

This last condition is the total differential of the equation . Notice, we can use this condition to obtain the result , which is what we obtained before.

 Total differentials are especially useful when you want to find how one variable is affected by more than one other variable in an implicit relationship. For example, consider the condition . Notice that we cannot solve for the variable  in this condition. However, let’s assume this equation implicitly defines  as a function of the variables  and . We can implicitly differentiate this condition as follows:



 Chain rule

 Constant coefficient rule

 Algebra

 Algebra

This last condition is the total differential. We can obtain the partial derivative  by setting . That is,

.

This partial derivative tells us how changes when  is held fixed (but  changes).

Similarly, the partial derivative



tells us how changes when  is held fixed (but  changes).

**Rules of Differentiation**

Involving Function of Different Variables

*Chain Rule*

$$z=f(y)$$

$$y=g\left(x\right)$$

=> $\frac{dz}{dx}=\frac{dz}{dy}\frac{dy}{dx}=f^{'}\left(x\right)\*g^{'}(x)$

$$∆x\rightarrow \left(via g \right)∆y\rightarrow \left(via f\right)∆z$$

$$z=f\left(y\right) y=g\left(x\right) x=h(w)$$

$$\frac{dz}{dw}=\frac{dz}{dy}\frac{dy}{dx}\frac{dx}{dw}=f^{'}\left(y\right)\*y'\left(x\right)\*h^{'}(w)$$

Ex. 1

if $z=3y^{2}$, where $y=2x+5$=>

$$\frac{dz}{dw}=\frac{dz}{dy}\frac{dy}{dx}=6y\*2=12y=12(2x+5)$$

Ex. 2

$z=y-3$, $y=x^{3}$=>

$$\frac{dz}{dx}=1\*3x^{2}=3x^{2}$$

Ex.3

$$z=\left(x^{2}+3x-2\right)^{17}$$

$z=y^{17}$ $y=x^{2}+3x-2$ --> the usefulness of the chain rule

$$\frac{dz}{dy}\frac{dy}{dx}=17y^{16}\left(2x+3\right)=17\left(x^{2}+3x-2\right)^{16}(2x+3)$$

Ex. 72

1) $TC=Q^{3}-5Q^{2}+12Q+75$

$$VC=Q^{3}-5Q^{2}+12Q$$

$$\frac{dVC}{dQ}=3Q^{2}-10Q+12$$

2) $AC=Q^{2}-4Q+174$

Find MC - ? $TC=AC\*Q=Q^{3}-4Q^{2}+174Q$

$$MC=\frac{dTC}{dQ}=3Q^{2}-8Q+174$$

$AC^{'}=2Q-4$=> $Q=2$

AC"=0 MC=AC => $Q^{2}-4Q+174=3Q^{2}-8Q+174$

$$2Q^{2}-4Q=0$$

$$2Q\left(Q-2\right)=0$$

Q=2

3) $TR=f(Q)$

$$Q=g(L)$$

Find $\frac{dTR}{dL}$

$$MRP\_{L}\rightarrow \frac{dTR}{dL}=\frac{dTR}{dQ}\frac{dQ}{dL}=f^{'}\left(Q\right)\*g^{'}\left(L\right)=MR\*MP\_{L}$$

$$MRP\_{L}=MR\*MP\_{L}$$