7.8 Cramer's rule for square linear systems

Let us consider the general system of linear equations when $m = n$. In this case it is called the square system of linear equations. Since this system can be represented as a matrix equation $AX = B$, we have $X = A^{-1}B$ if $\det A \neq 0$.

Theorem 1 *Any square system of linear equations has a unique solution if and only if the determinant of its coefficients matrix is not zero.*

Suppose that $C = (c_{ij})_{n\times n}$ with $c_{ij} = (-1)^{i+j}M_{ij}$. We know that $A^{-1} = \frac{1}{\det A}C^T$. Thus,

$$X = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix} = \frac{1}{\det A}\begin{pmatrix} c_{11} & c_{12} & ... & c_{1n} \\ c_{21} & c_{22} & ... & c_{21} \\ ... & ... & ... & ... \\ c_{n1} & c_{n2} & ... & c_{nn} \end{pmatrix}^T \begin{pmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{pmatrix}$$

$$= \frac{1}{\det A}\begin{pmatrix} c_{11} & c_{21} & ... & c_{n1} \\ c_{12} & c_{22} & ... & c_{n2} \\ ... & ... & ... & ... \\ c_{1n} & c_{2n} & ... & c_{nn} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{pmatrix} = \begin{pmatrix} c_{11}b_1 + c_{21}b_2 + \cdots + c_{n1}b_n \\ c_{12}b_1 + c_{22}b_2 + \cdots + c_{n2}b_n \\ ... \\ c_{1n}b_1 + c_{2n}b_2 + \cdots + c_{nn}b_n \end{pmatrix}.$$

Hence,

$$x_1 = \frac{c_{11}b_1 + c_{21}b_2 + \cdots + c_{n1}b_n}{\det A}$$
$$x_2 = \frac{c_{12}b_1 + c_{22}b_2 + \cdots + c_{n2}b_n}{\det A}$$
$$\cdots$$
$$x_n = \frac{c_{1n}b_1 + c_{2n}b_2 + \cdots + c_{nn}b_n}{\det A}$$

Let

$$\det A_1 = \begin{vmatrix} b_1 & a_{12} & ... & a_{1n} \\ b_2 & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ b_n & a_{n2} & ... & a_{nn} \end{vmatrix}.$$

This means that the matrix A_1 is a matrix formed from A by replacing its first column by the column of free terms. If we use "expansion by minors" with respect to this first column, we get

$$\det A_1 = \sum_{i=1}^{n} (-1)^{i+1} b_i M_{i1} = c_{11}b_1 + c_{21}b_2 + \cdots + c_{n1}b_n.$$

Therefore,

$$x_1 = \frac{\det A_1}{\det A}.$$

Similarly, we can write a general rule for all unknowns:

$$x_i = \frac{\det A_i}{\det A},$$

where A_i is the matrix formed by replacing the ith column of A with the column of free terms. This rule is known as *Cramer's rule.*

Example 1 *Solve the following system:* $\begin{cases} 3x_1 + x_2 = 1 \\ 5x_1 + 3x_2 = -1 \end{cases}.$

Solution 1

$$\det A = \begin{vmatrix} 3 & 1 \\ 5 & 3 \end{vmatrix} = 3 \cdot 3 - 1 \cdot 5 = 4$$

$$\det A_1 = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 1 \cdot 3 - 1 \cdot (-1) = 4$$

$$\det A_2 = \begin{vmatrix} 3 & 1 \\ 5 & -1 \end{vmatrix} = 3 \cdot (-1) - 1 \cdot 5 = -8$$

$$x_1 = \frac{\det A_1}{\det A} = \frac{4}{4} = 1$$

$$x_2 = \frac{\det A_2}{\det A} = \frac{-8}{4} = -2$$

Applications

Example 2 *A car dealer sells four types of cars through three stores located in Almaty, Astana, and Aktau. The following table presents the number of cars in each* *store:*

	Model 1	*Model 2*	*Model 3*	*Model 4*
Almaty	*10*	*8*	*5*	*10*
Astana	*5*	*3*	*5*	*7*
Aktau	*2*	*5*	*0*	*1*

The second table gives the prices and discount prices per each type of car (in thousands of dollars), respectively:

	Price	*Discount price*
Model 1	*30*	*26*
Model 2	*20*	*18*
Model 3	*50*	*40*
Model 4	*10*	*9*

Find the aggregate price-values of all cars in each store at the initial and discount prices.

Solution 2 *If we represent the tables as matrices and find their product, we answer the question. Thus, let*

$$A = \begin{pmatrix} 10 & 8 & 5 & 10 \\ 5 & 3 & 5 & 7 \\ 2 & 5 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 30 & 26 \\ 20 & 18 \\ 50 & 40 \\ 10 & 9 \end{pmatrix}$$

Then,

$$AB = \begin{pmatrix} 10 & 8 & 5 & 10 \\ 5 & 3 & 5 & 7 \\ 2 & 5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 30 & 26 \\ 20 & 18 \\ 50 & 40 \\ 10 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 10 \cdot 30 + 8 \cdot 20 + 5 \cdot 50 + 10 \cdot 10 & 10 \cdot 26 + 8 \cdot 18 + 5 \cdot 40 + 10 \cdot 9 \\ 5 \cdot 30 + 3 \cdot 20 + 5 \cdot 50 + 7 \cdot 10 & 5 \cdot 26 + 3 \cdot 18 + 5 \cdot 40 + 7 \cdot 9 \\ 2 \cdot 30 + 5 \cdot 20 + 0 \cdot 50 + 1 \cdot 10 & 2 \cdot 26 + 5 \cdot 18 + 0 \cdot 40 + 1 \cdot 9 \end{pmatrix}$$

$$= \begin{pmatrix} 810 & 694 \\ 530 & 443 \\ 170 & 151 \end{pmatrix}.$$

This means that the aggregate price-values of all cars in Almaty store are \$810000 and \$694000 at the initial and discount prices, respectively; in Astana store are \$530000 and \$443000; and in Aktau store are \$170000 and \$151000.

Example 3 *The equilibrium conditions for three related markets are given by*

$$\begin{cases} -p_3 = -11p_1 + p_2 + 6 \\ -p_1 - 2p_3 = 5 - 6p_2 \\ -2p_2 + 3p_3 = 4 + p_1 \end{cases}$$

Find out the equilibrium prices. Solve, using Cramer's rule.

Solution 3 *First step is to rewrite the system in the correct order:*

$$\begin{cases} 11p_1 - p_2 - p_3 = 6 \\ -p_1 + 6p_2 - 2p_3 = 5 \\ -p_1 - 2p_2 + 3p_3 = 4 \end{cases}$$

Now, we need to find $\det A$, $\det A_1$, $\det A_2$ *and* $\det A_3$.

$$\det A = \left| \begin{array}{ccc|cc} 11 & -1 & -1 & 11 & -1 \\ -1 & 6 & -2 & -1 & 6 \\ -1 & -2 & 3 & -1 & -2 \end{array} \right|$$

$$= (11 \cdot 6 \cdot 3 + (-1) \cdot (-2) \cdot (-1) + (-1) \cdot (-1) \cdot (-2))$$

$$-((-1) \cdot 6 \cdot (-1) + 11 \cdot (-2) \cdot (-2) + (-1) \cdot (-1) \cdot 3)$$

$$= (198 - 2 - 2) - (6 + 44 + 3) = 194 - 53 = 141$$

$$\det A_1 = \left| \begin{array}{ccc|cc} 6 & -1 & -1 & 6 & -1 \\ 5 & 6 & -2 & 5 & 6 \\ 4 & -2 & 3 & 4 & -2 \end{array} \right|$$

$$= (6 \cdot 6 \cdot 3 + (-1) \cdot (-2) \cdot 4 + (-1) \cdot 5 \cdot (-2)) - ((-1) \cdot 6 \cdot 4 + 6 \cdot (-2) \cdot (-2) + (-1) \cdot 5 \cdot 3)$$

$$= (108 + 8 + 10) - (-24 + 24 - 15) = 126 + 15 = 141$$

$$\det A_2 = \begin{vmatrix} 11 & 6 & -1 \\ -1 & 5 & -2 \\ -1 & 4 & 3 \end{vmatrix}\begin{matrix} 11 & 6 \\ -1 & 5 \\ -1 & 4 \end{matrix}\Bigg|$$

$$= (11\cdot 5\cdot 3+6\cdot(-2)\cdot(-1)+(-1)\cdot(-1)\cdot 4)-((-1)\cdot 5\cdot(-1)+11\cdot(-2)\cdot 4+6\cdot(-1)\cdot 3)$$

$$= (165 + 12 + 4) - (5 - 88 - 18) = 181 + 101 = 282$$

$$\det A_3 = \begin{vmatrix} 11 & -1 & 6 \\ -1 & 6 & 5 \\ -1 & -2 & 4 \end{vmatrix}\begin{matrix} 11 & -1 \\ -1 & 6 \\ -1 & -2 \end{matrix}\Bigg|$$

$$= (11\cdot 6\cdot 4+(-1)\cdot 5\cdot(-1)+6\cdot(-1)\cdot(-2))-(6\cdot 6\cdot(-1)+11\cdot 5\cdot(-2)+(-1)\cdot(-1)\cdot 4)$$

$$= (264 + 5 + 12) - (-36 - 110 + 4) = 281 + 142 = 423$$

$$p_1 = \frac{\det A_1}{\det A} = \frac{141}{141} = 1$$

$$p_2 = \frac{\det A_2}{\det A} = \frac{282}{141} = 2$$

$$p_3 = \frac{\det A_3}{\det A} = \frac{423}{141} = 3$$

Input-output analysis

One of applications of matrices to economics is called Leontief input-output analysis. A typical economy depends on a number of sectors. Each sector requires input from other sectors to produce its output. Moreover, there can be some outside demands. Input-output analysis helps to determine output levels of each sectors that meet various levels of final demands. Let us consider an example.

Example 4 *Let some economy depend on two industries: coal mining company and coal-fired power plant that produces electricity. Suppose that the production of one tenge worth of coal requires 0.1 tenge worth of coal and 0.4 tenge worth of electricity; the production of one tenge worth of electricity requires 0.7 tenge worth of coal and 0.2 tenge worth of electricity. If the demand from the outside consumers is 1 billion tenge for coal and 7 billion tenge for electricity, how much coal and electricity should be produced to satisfy the final demand?*

Solution 4 *Let x_1 be the total output from the coal mining company and x_2 be the total output from the coal-fired power plant. Moreover, $d_1 = 1$ (billion) be the outside coal demand and $d_2 = 7$ (billion) be the outside electricity demand. Combining the internal demands with the outside demands, we get:*

$$x_1 = 0.1x_1 + 0.7x_2 + d_1$$

$$x_2 = 0.4x_1 + 0.2x_2 + d_2$$

If we rewrite the above system as a matrix equation, we have:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.1 & 0.7 \\ 0.4 & 0.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

or

$$X = MX + D,$$

where M is called the technology matrix.

We need to resolve the equation with respect to X:

$$X - MX = D$$

$$(I - M)X = D$$

$$D = (I - M)^{-1}D$$

Then,

$$I - M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.1 & 0.7 \\ 0.4 & 0.2 \end{pmatrix} = \begin{pmatrix} 0.9 & -0.3 \\ -0.6 & 0.8 \end{pmatrix}$$

$$\det(I - M) = \begin{vmatrix} 0.9 & -0.3 \\ -0.6 & 0.8 \end{vmatrix} = 0.72 - 0.18 = 0.54$$

$$(I - M)^{-1} = \frac{1}{0.54} \begin{pmatrix} 0.8 & 0.3 \\ 0.6 & 0.9 \end{pmatrix}$$

$$X = \frac{1}{0.54} \begin{pmatrix} 0.8 & 0.3 \\ 0.6 & 0.9 \end{pmatrix} \begin{pmatrix} 1 \\ 7 \end{pmatrix} = \frac{1}{0.54} \begin{pmatrix} 2.9 \\ 6.9 \end{pmatrix} = \begin{pmatrix} \approx 5.37 \\ \approx 12.78 \end{pmatrix}$$

This means that the coal mining company must have a coal output of 5.37 *billion tenge, and the coal-fired power plant must have an electricity output of* 12.78 *billion tenge to meet both demands.*