7.5 Matrix equations

Summing up all information concerning matrices, we list their basic properties:

Addition properties

1. $A + B = B + A$ (commutative);

2. $(A + B) + C = A + (B + C)$ (associative);

3. $A + 0 = 0 + A = A$;

4. $A - A = -A + A = 0$.

Multiplication properties

1. $A(BC) = (AB)C$ (associative);

2. $AI = IA = A$;

3. $AA^{-1} = A^{-1}A = I$.

Combined properties

1. $A(B + C) = AB + AC$;

2. $(B + C)A = BA + CA$.

Equality properties

1. If $A = B$, then $A + C = B + C$;

2. If $A = B$, then $AC = BC$;

3. If $A = B$, then $CA = CB$.

We use these properties for the solution of matrix equations.

Example 1 *Given an $n \times n$ matrix A and an $n \times m$ matrix B. Solve $AX = B$ for X, where X is an $n \times m$ matrix.*

Solution 1

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

Example 2 *Given an $n \times n$ matrix A and an $m \times n$ matrix B. Solve $XA = B$ for X, where X is an $m \times n$ matrix.*

Solution 2

$$XA = B$$

$$XAA^{-1} = BA^{-1}$$

$$XI = BA^{-1}$$

$$X = BA^{-1}$$

Example 3 *Given an $n \times n$ matrix A and an $n \times m$ matrix B. Solve $AX + X = B$ for X, where X is an $n \times m$ matrix.*

Solution 3

$$AX + X = B$$

$$(A + I)X = B$$

$$(A + I)^{-1}(A + I)X = (A + I)^{-1}B$$

$$IX = (A + I)^{-1}B$$

$$X = (A + I)^{-1}B$$

Example 4 $A = \begin{pmatrix} 4 & 5 \\ -1 & -2 \end{pmatrix}$ *and* $B = \begin{pmatrix} -1 & -3 \\ 5 & 1 \end{pmatrix}$. *Solve $X - AX = B$ for X.*

Solution 4 *First step is to resolve the given equation with resect to X:*

$$X - AX = B$$

$$(I - A)X = B$$

$$(I - A)^{-1}(I - A)X = (I - A)^{-1}B$$

$$X = (I - A)^{-1}B$$

So, we need to find $(I - A)^{-1}$:

$$I - A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 5 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -3 & -5 \\ 1 & 3 \end{pmatrix}$$

$$\det(I-A)=\begin{vmatrix} -3 & -5 \\ 1 & 3 \end{vmatrix} = -9+5=-4$$

$$(I-A)^{-1}=\frac{1}{\det A}\begin{pmatrix} M_{11} & -M_{12} \\ -M_{21} & M_{22} \end{pmatrix}^T = -\frac{1}{4}\begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix}^T = -\frac{1}{4}\begin{pmatrix} 3 & 5 \\ -1 & -3 \end{pmatrix}$$

Thus,

$$X=-\frac{1}{4}\begin{pmatrix} 3 & 5 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} -1 & -3 \\ 5 & 1 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} 22 & -4 \\ -14 & 0 \end{pmatrix} = \begin{pmatrix} -5.5 & 1 \\ 3.5 & 0 \end{pmatrix}.$$

7.6 System of linear equations

A system of m equations with n unknowns (an $m \times n$ system) can be written in the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

where a_{ij} is called the coefficient of the unknown x_j, and b_i is called the constant (or free term) of the ith equation.

The system can be rewritten as a matrix equation of the form $AX = B$, where

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad X=\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \text{ and } B=\begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$$

are called the matrix of coefficients, the matrix of unknowns and the matrix of constants (or free terms), respectively.

Moreover, if the coefficient matrix is augmented by the constant matrix, then we get the augmented matrix of the system:

$$\overline{A}=\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right).$$

The linear system must have

(A) exactly one solution (then it is called consistent and independent);

(B) infinitely many solutions (then it is called consistent and dependent);

(C) no solution (then it is called inconsistent).

There are no other possibilities.

Remark 1 *"Independent" means that each equation gives new information. Otherwise they are "dependent".*

These three cases have a geometric illustration when the system consists of two equations with two unknowns.

Example 5 *Solve the following linear systems:*

(A) $\begin{cases} 3x_1 + x_2 = 1 \\ 5x_1 + 3x_2 = -1 \end{cases}$

(B) $\begin{cases} 3x_1 + x_2 = 1 \\ -6x_1 - 2x_2 = -2 \end{cases}$

(C) $\begin{cases} 3x_1 + x_2 = 1 \\ 3x_1 + x_2 = -1 \end{cases}$

Solution 5 *(A) Let us use the substitution technique:*

$$x_2 = -3x_1 + 1$$

$$3(-3x_1 + 1) + 5x_1 = -1$$

$$-9x_1 + 3 + 5x_1 = -1$$

$$-4x_1 = -4$$

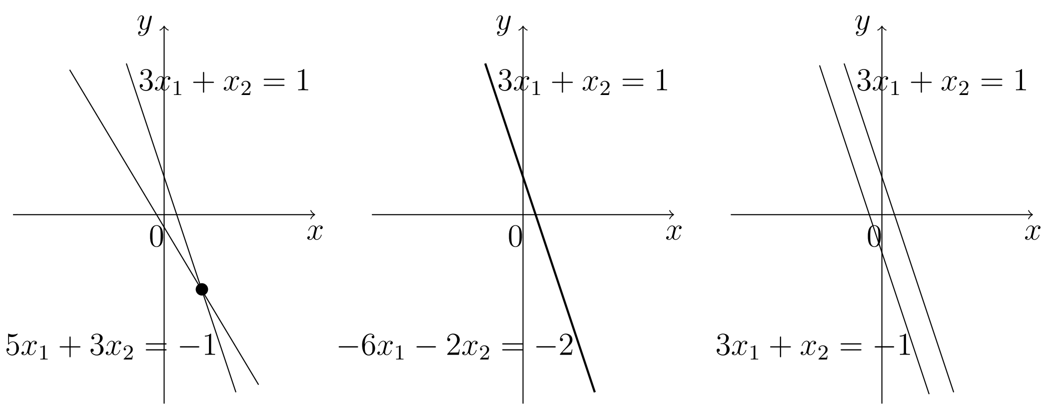
$$x_1 = 1$$

$$x_2 = -2$$

The graphs of both equations are straight lines $x_2 = -3x_1 + 1$ *and* $x_2 = -\frac{5}{3}x_1 - \frac{1}{3}$. *They are shown in Figure 32 (A). They intersect at the point (1;-2), the solution of* *system.*

(B) Those equations are "dependent", because they are really the same equations. So, the second equation gives no new information. They have the same graph, as shown in Figure 32 (B). Thus, all points of the straight line $x_2 = -3x_1 + 1$ *satisfy the system. This means that the system has infinitely many solutions, namely,* x_1 *is any real number, while* $x_2 = -3x_1 + 1$.

(C) The graphs of these equations are parallel lines $x_2 = -3x_1 + 1$ *and* $x_2 = -3x_1 - 1$ *(each has slope* -3*), as shown in Figure 32 (C). Therefore, the system has no solution. However, it is not necessary to draw the graphs to discover this fact. If we subtract the second equation from the first, we get the false statement:* $0 = 2$. *This means that the system is inconsistent.*



(A) Lines intersect at one point: exactly one solution *(A) Lines coincide: infinitely many solutions* *(A) Lines are parallel: no solution*

Figure 32

7.7 Gaussian elimination

Definition 1 *A matrix A is called an echelon matrix, if the following two conditions hold:*

1. All zero rows, if any, are at the bottom of the matrix;

2. Each leading nonzero entry in a row is to the right of the leading nonzero entry in the preceding row.

For example,

$$A = \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 9 & 0 & 7 \\ 0 & 0 & 0 & 3 & 4 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 5 & 7 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Elementary row operations

1. Interchange rows R_i and R_j or "$R_i \leftrightarrow R_j$";

2. Replace R_i by kR_i ($k \neq 0$) or "$kR_i \to R_i$";

3. Replace R_j by $kR_i + R_j$ or "$kR_i + R_j \to R_j$".

Definition 2 *A matrix A is said to be row equivalent to a matrix B, written $A \sim B$, if B can be obtained from A by a sequence of elementary row operations.*

Algorithm of Gaussian elimination: The input is any matrix A and the output is an echelon form of A.

Step 1. Find the first column with a nonzero entry. Let j denote this column.

(a) arrange so that $a_{1j} \neq 0$.

(b) Use a_{1j} as a pivot to obtain 0's below a_{1j}: replace R_i by $kR_1 + R_i$, where $k = -\frac{a_{ij}}{a_{1j}}$.

Step 2. Repeat Step 1 with the submatrix formed by all the rows excluding the first row.

Step 3. Continue the above process until a submatrix has only zero rows.

Remark 2 *Thus, by elementary row operations any matrix can be reduced to echelon form.*

Example 6 *Reduce* $A = \begin{pmatrix} 1 & 2 & -3 & 1 & 2 \\ 2 & 4 & -4 & 6 & 10 \\ 3 & 6 & -6 & 9 & 13 \end{pmatrix}$ *to echelon form.*

Solution 6 $\begin{pmatrix} 1 & 2 & -3 & 1 & 2 \\ 2 & 4 & -4 & 6 & 10 \\ 3 & 6 & -6 & 9 & 13 \end{pmatrix} \quad -2R_1 + R_2 \to R_2 \quad \textit{and} \quad -3R_1 + R_3 \to R_3$

$$\sim \begin{pmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 3 & 6 & 7 \end{pmatrix} \quad \tfrac{1}{2}R_2 \to R_2$$

$$\sim \begin{pmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 6 & 7 \end{pmatrix} \quad -3R_2 + R_3 \to R_3$$

$$\sim \begin{pmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

Theorem 1 *If augmented matrices of two systems of linear equations are row equivalent, then these systems have the same solution.*

Solving a system of linear equations using Gaussian elimination

Step 1 (forward elimination). Step-by-step reduction of the augmented matrix of the given system into an equivalent echelon form matrix.

Step 2 (backward elimination). Step-by-step back-substitution to find a solution of the simpler system constructed from the obtained equivalent echelon form matrix.

Remark 3 *From Step 2 it is obvious that a system has a solution if and only if an echelon form of its augmented matrix does not have a row of the form* $(0, 0, 0, ..., b)$ *with* $b \neq 0$.

Example 7 *Solve the system* $\begin{cases} x_1 + x_2 - 2x_3 + 4x_4 = 5 \\ 2x_1 + 2x_2 - 3x_3 + x_4 = 3 \\ 3x_1 + 3x_2 - 4x_3 - 2x_4 = 1 \end{cases}$.

Solution 7

$$\overline{A} = \left(\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 2 & 2 & -3 & 1 & 3 \\ 3 & 3 & -4 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 2 & -14 & -14 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

That is the given system has the same solution as the system:

$$\begin{cases} x_1 + x_2 - 2x_3 + 4x_4 = 5 \\ \qquad\qquad\quad x_3 - 7x_4 = -7 \end{cases}.$$

From the last system by back-substitution we have that $x_3 = -7 + 7x_4$ *and* $x_1 = -9 - x_2 + 10x_4$. *Thus, if* a *and* b *are any real numbers, then*

$$x_1 = -9 - a + 10b$$
$$x_2 = a$$
$$x_3 = -7 + 7a$$
$$x_4 = b$$

are solutions of the given system.

Example 8 *Solve the system* $\begin{cases} x_1 + x_2 - 2x_3 + 3x_4 = 4 \\ 2x_1 + 3x_2 + 3x_3 - x_4 = 3 \\ 5x_1 + 7x_2 + 4x_3 + x_4 = 5 \end{cases}.$

Solution 8

$$\overline{A} = \left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & 1 & 5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 2 & 14 & -14 & -15 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right).$$

That is the $0 \cdot x_4 = -5$. *Thus, the system has no solution.*

Example 9 *Solve the system* $\begin{cases} x_1 + 2x_2 + x_3 = 3 \\ 2x_1 + 5x_2 - x_3 = -4 \\ 3x_1 - 2x_2 - x_3 = 5 \end{cases}.$

Solution 9

$$\overline{A} = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 5 & -1 & -4 \\ 3 & -2 & -1 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & -8 & -4 & -4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & -28 & -84 \end{array}\right).$$

That is the given system has the same solution as the system:

$$\begin{cases} x_1 + 2x_2 + x_3 = 3 \\ \qquad x_2 - 3x_3 = -10 \\ \qquad\quad -28x_3 = -84 \end{cases}.$$

From the last system by back-substitution we get that

$$x_1 = 2$$
$$x_2 = -1$$
$$x_3 = 3$$

is a solution of the given system.