Matrix equations

Summing up all information concerning matrices, we list their basic properties:

Addition properties

1. $A + B = B + A$ (commutative);

2. $(A + B) + C = A + (B + C)$ (associative);

3. $A + 0 = 0 + A = A$;

4. $A - A = -A + A = 0$.

Multiplication properties

1. $A(BC) = (AB)C$ (associative);

2. $AI = IA = A$;

3. $AA^{-1} = A^{-1}A = I$.

Combined properties

1. $A(B + C) = AB + AC$;

2. $(B + C)A = BA + CA$.

Equality properties

1. If $A = B$, then $A + C = B + C$;

2. If $A = B$, then $AC = BC$;

3. If $A = B$, then $CA = CB$.

We use these properties for the solution of matrix equations.

Example 1 *Given an $n \times n$ matrix A and an $n \times m$ matrix B. Solve $AX = B$ for X, where X is an $n \times m$ matrix.*

Solution 1

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

Example 2 *Given an $n \times n$ matrix A and an $m \times n$ matrix B. Solve $XA = B$ for X, where X is an $m \times n$ matrix.*

Solution 2

$$XA = B$$

$$XAA^{-1} = BA^{-1}$$

$$XI = BA^{-1}$$

$$X = BA^{-1}$$

Example 3 *Given an $n \times n$ matrix A and an $n \times m$ matrix B. Solve $AX + X = B$ for X, where X is an $n \times m$ matrix.*

Solution 3

$$AX + X = B$$

$$(A + I)X = B$$

$$(A + I)^{-1}(A + I)X = (A + I)^{-1}B$$

$$IX = (A + I)^{-1}B$$

$$X = (A + I)^{-1}B$$

Example 4 $A = \begin{pmatrix} 4 & 5 \\ -1 & -2 \end{pmatrix}$ *and* $B = \begin{pmatrix} -1 & -3 \\ 5 & 1 \end{pmatrix}$. *Solve $X - AX = B$ for X.*

Solution 4 *First step is to resolve the given equation with resect to X:*

$$X - AX = B$$

$$(I - A)X = B$$

$$(I - A)^{-1}(I - A)X = (I - A)^{-1}B$$

$$X = (I - A)^{-1}B$$

So, we need to find $(I - A)^{-1}$:

$$I - A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 5 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -3 & -5 \\ 1 & 3 \end{pmatrix}$$

$$\det(I-A)=\begin{vmatrix}-3 & -5\\ 1 & 3\end{vmatrix}=-9+5=-4$$

$$(I-A)^{-1}=\frac{1}{\det A}\begin{pmatrix}M_{11} & -M_{12}\\ -M_{21} & M_{22}\end{pmatrix}^T=-\frac{1}{4}\begin{pmatrix}3 & -1\\ 5 & -3\end{pmatrix}^T=-\frac{1}{4}\begin{pmatrix}3 & 5\\ -1 & -3\end{pmatrix}$$

Thus,

$$X=-\frac{1}{4}\begin{pmatrix}3 & 5\\ -1 & -3\end{pmatrix}\begin{pmatrix}-1 & -3\\ 5 & 1\end{pmatrix}=-\frac{1}{4}\begin{pmatrix}22 & -4\\ -14 & 0\end{pmatrix}=\begin{pmatrix}-5.5 & 1\\ 3.5 & 0\end{pmatrix}.$$

Gaussian elimination

Definition 1 *A matrix A is called an echelon matrix, if the following two conditions hold:*

1. All zero rows, if any, are at the bottom of the matrix;

2. Each leading nonzero entry in a row is to the right of the leading nonzero entry in the preceding row.

For example,

$$A=\begin{pmatrix}0&2&3&4&5&9&0&7\\0&0&0&3&4&1&2&5\\0&0&0&0&0&5&7&2\\0&0&0&0&0&0&8&6\\0&0&0&0&0&0&0&0\end{pmatrix}$$

Elementary row operations

1. Interchange rows R_i and R_j or "$R_i \leftrightarrow R_j$";
2. Replace R_i by kR_i $(k \neq 0)$ or "$kR_i \to R_i$";
3. Replace R_j by $kR_i + R_j$ or "$kR_i + R_j \to R_j$".

Definition 2 *A matrix A is said to be row equivalent to a matrix B, written $A \sim B$, if B can be obtained from A by a sequence of elementary row operations.*

Algorithm of Gaussian elimination: The input is any matrix A and the output is an echelon form of A.

Step 1. Find the first column with a nonzero entry. Let j denote this column.

(a) arrange so that $a_{1j} \neq 0$.

(b) Use a_{1j} as a pivot to obtain 0's below a_{1j}: replace R_i by $kR_1 + R_i$, where $k = -\frac{a_{ij}}{a_{1j}}$.

Step 2. Repeat Step 1 with the submatrix formed by all the rows excluding the first row.

Step 3. Continue the above process until a submatrix has only zero rows.

Remark 1 *Thus, by elementary row operations any matrix can be reduced to echelon form.*

Example 5 *Reduce* $A = \begin{pmatrix} 1 & 2 & -3 & 1 & 2 \\ 2 & 4 & -4 & 6 & 10 \\ 3 & 6 & -6 & 9 & 13 \end{pmatrix}$ *to echelon form.*

Solution 5

$$\begin{pmatrix} 1 & 2 & -3 & 1 & 2 \\ 2 & 4 & -4 & 6 & 10 \\ 3 & 6 & -6 & 9 & 13 \end{pmatrix} \quad -2R_1 + R_2 \to R_2 \quad \textit{and} \quad -3R_1 + R_3 \to R_3$$

$$\sim \begin{pmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 3 & 6 & 7 \end{pmatrix} \quad \tfrac{1}{2}R_2 \to R_2$$

$$\sim \begin{pmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 6 & 7 \end{pmatrix} \quad -3R_2 + R_3 \to R_3$$

$$\sim \begin{pmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

Inverse of using Elementary Row Operations

This method is as follows: apply Elementary Row Operations to a matrix A until it turns into the identity matrix I, and by also doing the changes to an identity matrix it turns into the inverse A^{-1}. Thus,

$$(A|I) \sim \left(I|A^{-1}\right).$$

Example 6 *Find* A^{-1} *if* $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -5 & 3 \\ 2 & 7 & -1 \end{pmatrix}$.

Solution 6 $\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & -5 & 3 & 0 & 1 & 0 \\ 2 & 7 & -1 & 0 & 0 & 1 \end{array}\right)$ $\quad -3R_1+R_2 \to R_2 \;\; and \;\; -2R_1+R_3 \to R_3$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -11 & 0 & -3 & 1 & 0 \\ 0 & 3 & -3 & -2 & 0 & 1 \end{array}\right) \quad -\tfrac{1}{11}R_2 \to R_2 \;\; and \;\; \tfrac{1}{3}R_3 \to R_3$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{11} & -\frac{1}{11} & 0 \\ 0 & 1 & -1 & -\frac{2}{3} & 0 & \frac{1}{3} \end{array}\right) \quad R_3+R_1 \to R_1 \;\; and \;\; (-1)R_2+R_3 \to R_3$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{3}{11} & -\frac{1}{11} & 0 \\ 0 & 0 & -1 & -\frac{31}{33} & \frac{1}{11} & \frac{1}{3} \end{array}\right) \quad (-3)R_2+R_1 \to R_1 \;\; and \;\; (-1)R_3 \to R_3$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{16}{33} & \frac{3}{11} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{3}{11} & -\frac{1}{11} & 0 \\ 0 & 0 & 1 & \frac{31}{33} & -\frac{1}{11} & -\frac{1}{3} \end{array}\right).$$

Thus, $A^{-1} = \begin{pmatrix} -\frac{16}{33} & \frac{3}{11} & \frac{1}{3} \\ \frac{3}{11} & -\frac{1}{11} & 0 \\ \frac{31}{33} & -\frac{1}{11} & -\frac{1}{3} \end{pmatrix}$.