7 Linear algebra

7.1 Matrices

Definition 1 *An* $m \times n$ *matrix is a rectangular table with* m *rows and* n *columns of elements* a_{ij}*,* $i = 1, 2, ..., m$*,* $j = 1, 2, ..., n$*, written within brackets*

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{21} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}.$$

The expression $m \times n$ *is called the size of the matrix.*

Remark 1 *It is important to note that the number of rows if always given first.*

Remark 2 *Each element of a matrix is uniquely identified by its row and column position. This position is denoted by double subscript notation* a_{ij}*, where* i *is the row* R_i *and* j *is the column* C_j *containing* a_{ij}*.*

Definition 2 *A matrix with* n *rows and* n *columns is called a square matrix of order* n*.*

Definition 3 *A matrix with only 1 column is called a column matrix or column vector, and a matrix with only 1 row is called a row matrix or row vector.*

Definition 4 *The principal (main) diagonal of a matrix consists of the elements* a_{11}*,* a_{22}*,* a_{33}*, ...*

Definition 5 *A matrix consisting only of zeros is called a zero matrix and is denoted* O*.*

Definition 6 *An identity matrix* I *is a square matrix that has 1 on the principal diagonal and 0 elsewhere. For example, the identity matrices* 2×2*,* 3×3 *and* 4×4

are

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \textit{and} \quad I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

respectively.

Definition 7 *The transpose of a matrix A is a new matrix A^T formed by interchanging the rows and columns of A.*

7.2 Operations with matrices

1. Sum and difference of two matrices

Remark 3 *The sum and difference operations are not defined for matrices of different sizes.*

The sum of two matrices is a matrix with elements that are the sum of the corresponding elements of two given matrices. Similarly, the difference of two matrices is a matrix with elements that are the difference of the corresponding elements of two given matrices.

Thus, suppose that $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$. If $C = (c_{ij})_{m\times n} = A + B$ and $D = (d_{ij})_{m\times n} = A - B$, then

$$c_{ij} = a_{ij} + b_{ij}$$

and

$$d_{ij} = a_{ij} - b_{ij}.$$

Remark 4 *Matrix summation is commutative:*

$$A + B = B + A.$$

Example 1 $A = \begin{pmatrix} 2 & -1 & 0 & 7 \\ 4 & 6 & -7 & -3 \end{pmatrix}$ *and* $B = \begin{pmatrix} 6 & 2 & -5 & 5 \\ 2 & 1 & -2 & 1 \end{pmatrix}$. *Find $A + B$ and $A - B$.*

Solution 1

$$A+B=\begin{pmatrix} 2+6 & -1+2 & 0+(-5) & 7+5 \\ 4+2 & 6+1 & -7+(-2) & -3+1 \end{pmatrix}$$

$$=\begin{pmatrix} 8 & 1 & -5 & 12 \\ 6 & 7 & -9 & -2 \end{pmatrix}.$$

$$A-B=\begin{pmatrix} 2-6 & -1-2 & 0-(-5) & 7-5 \\ 4-2 & 6-1 & -7-(-2) & -3-1 \end{pmatrix}$$

$$=\begin{pmatrix} -4 & -3 & 5 & 2 \\ 2 & 5 & -5 & -4 \end{pmatrix}.$$

2. Multiplication of matrix by number

The product of a matrix A by a number k is a matrix kA formed by multiplying each element a_{ij} by k.

Example 2 $A=\begin{pmatrix} 2 & -1 & 0 & 7 \\ 4 & 6 & -7 & -3 \end{pmatrix}$ *and* $k=10$. *Find* kA.

Solution 2

$$kA=10\begin{pmatrix} 2 & -1 & 0 & 7 \\ 4 & 6 & -7 & -3 \end{pmatrix}=\begin{pmatrix} 20 & -10 & 0 & 70 \\ 40 & 60 & -70 & -30 \end{pmatrix}.$$

3. Multiplication of two matrices

Remark 5 *The product of two matrices AB is not defined if the number of columns in A is not equal to the number of rows in B.*

The product of two matrices A and B is a matrix with an element in the ith row and jth column obtained by multiplying the corresponding entries of the ith row of A and jth column of B (first by first, second by second, etc.) and adding the result.

Thus, suppose that $A=(a_{ik})_{m\times p}$ and $B=(b_{kj})_{p\times n}$. If $C=(c_{ij})_{m\times n}=AB$, then

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{ik}b_{kj}.$$

Remark 6 *Matrix product is not commutative:*

$$AB \neq BA.$$

Example 3 $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$. *Find* AB *and* BA.

Solution 3

$$AB = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$$

is not defined because 3 *columns in* A *is not equal to* 2 *rows in* B.

$$BA = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 3 \cdot 3 & 1 \cdot 2 + 3 \cdot 1 & 1 \cdot 1 + 3 \cdot 0 \\ 1 \cdot 1 + 2 \cdot 3 & 1 \cdot 2 + 2 \cdot 1 & 1 \cdot 1 + 2 \cdot 0 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 5 & 1 \\ 7 & 4 & 1 \end{pmatrix}.$$