Area between two curves

Theorem 1 *If f and g are continuous functions such that $f(x) \geq g(x)$ over the interval $[a;b]$, then the area of a region bounded by $f(x)$ and $g(x)$ over $[a;b]$ is given by the formula*

$$S = \int_a^b (f(x) - g(x))\, dx.$$

Example 1 *Find the area bounded by $f(x) = x^2 + 2$ and $g(x) = 1 - x^2$ over $[0;1]$.*

Solution 1 *First, we need to sketch the area (Figure 1).*

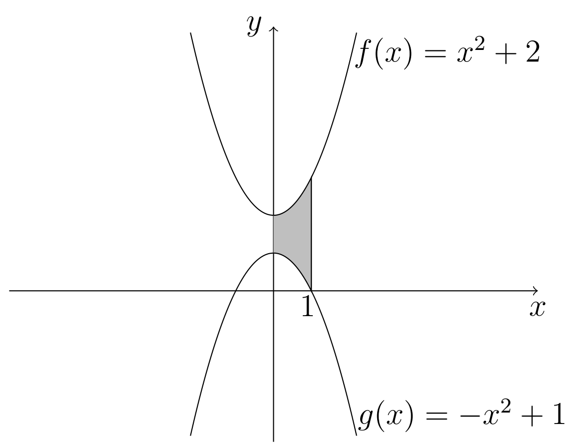

Figure 1

It is obvious that $f(x) \geq g(x)$, therefore

$$S = \int_0^1 \left((x^2 + 2) - (1 - x^2)\right) dx = \int_0^1 \left(2x^2 + 1\right) dx = \frac{2x^3}{3} + x \,\Big|_0^1 = \frac{2}{3} + 1 = \frac{5}{3}.$$

Example 2 *Find the area bounded by $f(x) = \sqrt{x}$ and $g(x) = x^2$.*

Solution 2 *The statement of the problem does not include any interval. It means that we need to find points of intersection of f and g, that is equivalent to the solution of the equation $f(x) = g(x)$ or $\sqrt{x} = x^2$. That gives $x_1 = 0$ and $x_2 = 1$.*

Second, we need to sketch the area (Figure 2).

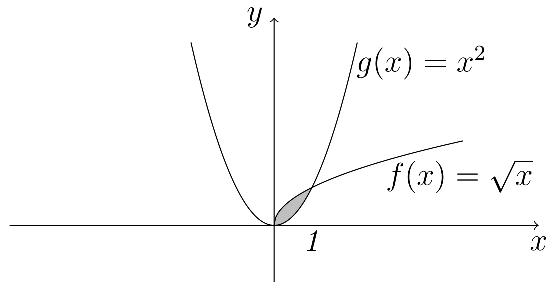

Figure 2

It is obvious that $f(x) \geq g(x)$ *for* $[0; 1]$*, therefore*

$$S = \int_0^1 \left(\sqrt{x} - x^2\right) dx = \frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3} \Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

Example 3 *Find the area bounded by* $f(x) = (x+3)^2 - 3$ *and* $g(x) = 2x + 6$ *over* $[-5; 0]$.

Solution 3 *Let us find points of intersection of* f *and* g*, that is equivalent to the solution of the equation* $f(x) = g(x)$.

$$(x+3)^2 - 3 = 2x + 6$$

$$x^2 + 6x + 9 - 3 - 2x - 6 = 0$$

$$x^2 + 4x = 0$$

$$x_1 = 0, \ \ x_2 = -4.$$

These points help us to sketch the area (Figure 3).

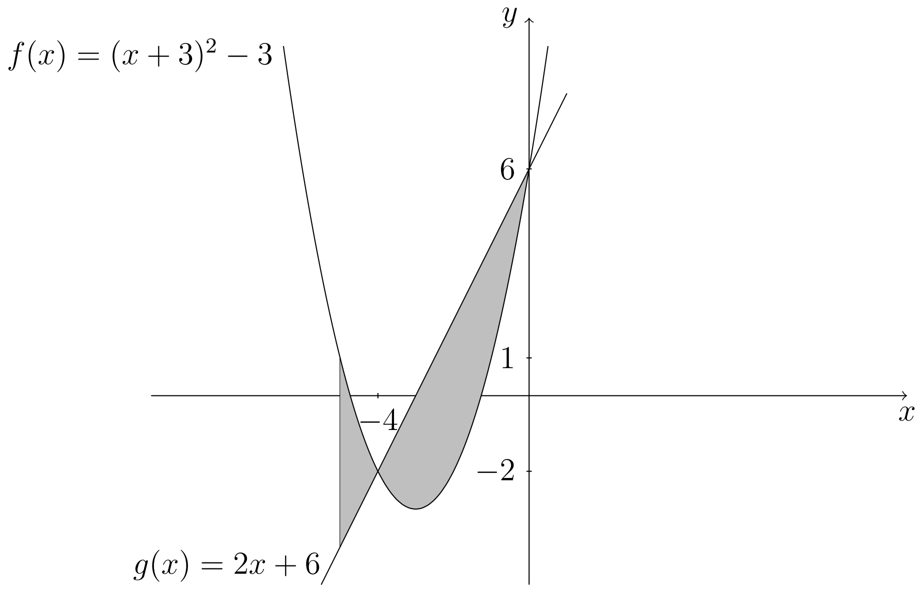

Figure 3

Examining the graph, we see that $f(x) \geq g(x)$ *over* $[-5;-4]$ *and* $g(x) \geq f(x)$ *over* $[-4;0]$. *Thus, two integrals are needed to calculate the total area.*

$$S_1 = \int_{-5}^{-4} (f(x) - g(x))\,dx = \int_{-5}^{-4} \big(((x+3)^2 - 3) - (2x+6)\big)\,dx$$

$$= \int_{-5}^{-4} \big(x^2 + 6x + 9 - 3 - 2x - 6\big)\,dx = \int_{-5}^{-4} \big(x^2 + 4x\big)\,dx = \frac{x^3}{3} + 2x^2 \,\Big|_{-5}^{-4}$$

$$= \left(\frac{(-4)^3}{3} + 2(-4)^2\right) - \left(\frac{(-5)^3}{3} + 2(-5)^2\right) = -\frac{64}{3} + 32 + \frac{125}{3} - 50$$

$$= \frac{61}{3} - 18 = \frac{61}{3} - \frac{54}{3} = \frac{7}{3}.$$

$$S_2 = \int_{-4}^{0} (g(x) - f(x))\,dx = \int_{-4}^{0} \big((2x+6) - ((x+3)^2 - 3)\big)\,dx$$

$$= \int_{-4}^{0} \big(2x + 6 - x^2 - 6x - 9 + 3\big)\,dx = \int_{-4}^{0} \big(-x^2 - 4x\big)\,dx = -\frac{x^3}{3} - 2x^2 \,\Big|_{-4}^{0}$$

$$= 0 - \left(-\frac{(-4)^3}{3} - 2(-4)^2\right) = -\frac{64}{3} + 32 = -\frac{64}{3} + \frac{96}{3} = \frac{32}{3}.$$

The total area between the two graphs is

$$S = S_1 + S_2 = \frac{7}{3} + \frac{32}{3} = \frac{39}{3} = 13.$$