6.7 Area between two curves

Theorem 1 *If f and g are continuous functions such that $f(x) \geq g(x)$ over the interval $[a;b]$, then the area of a region bounded by $f(x)$ and $g(x)$ over $[a;b]$ is given by the formula*

$$S = \int_a^b (f(x) - g(x))\, dx.$$

Example 1 *Find the area bounded by $f(x) = x^2 + 2$ and $g(x) = 1 - x^2$ over $[0;1]$.*

Solution 1 *First, we need to sketch the area (Figure 25).*

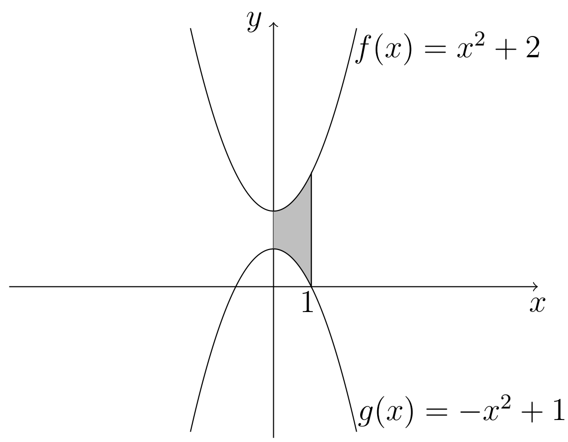

Figure 25

It is obvious that $f(x) \geq g(x)$, therefore

$$S = \int_0^1 \left((x^2+2) - (1-x^2)\right) dx = \int_0^1 \left(2x^2+1\right) dx = \frac{2x^3}{3} + x \,\Big|_0^1 = \frac{2}{3} + 1 = \frac{5}{3}.$$

Example 2 *Find the area bounded by $f(x) = \sqrt{x}$ and $g(x) = x^2$.*

Solution 2 *The statement of the problem does not include any interval. This means that we need to find points of intersection of f and g, that is equivalent to the solution of the equation $f(x) = g(x)$ or $\sqrt{x} = x^2$. That gives $x_1 = 0$ and $x_2 = 1$.*

Second, we need to sketch the area (Figure 26).

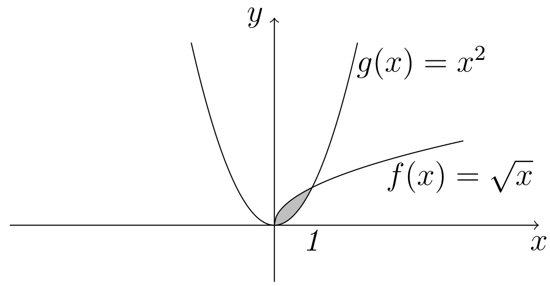

Figure 26

It is obvious that $f(x) \geq g(x)$ *for* $[0;1]$, *therefore*

$$S = \int_0^1 \left(\sqrt{x} - x^2\right) dx = \frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3} \Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

Example 3 *Find the area bounded by* $f(x) = (x+3)^2 - 3$ *and* $g(x) = 2x + 6$ *over* $[-5;0]$.

Solution 3 *Let us find points of intersection of* f *and* g, *that is equivalent to the solution of the equation* $f(x) = g(x)$.

$$(x+3)^2 - 3 = 2x + 6$$

$$x^2 + 6x + 9 - 3 - 2x - 6 = 0$$

$$x^2 + 4x = 0$$

$$x_1 = 0, \ \ x_2 = -4.$$

These points help us to sketch the area (Figure 27).

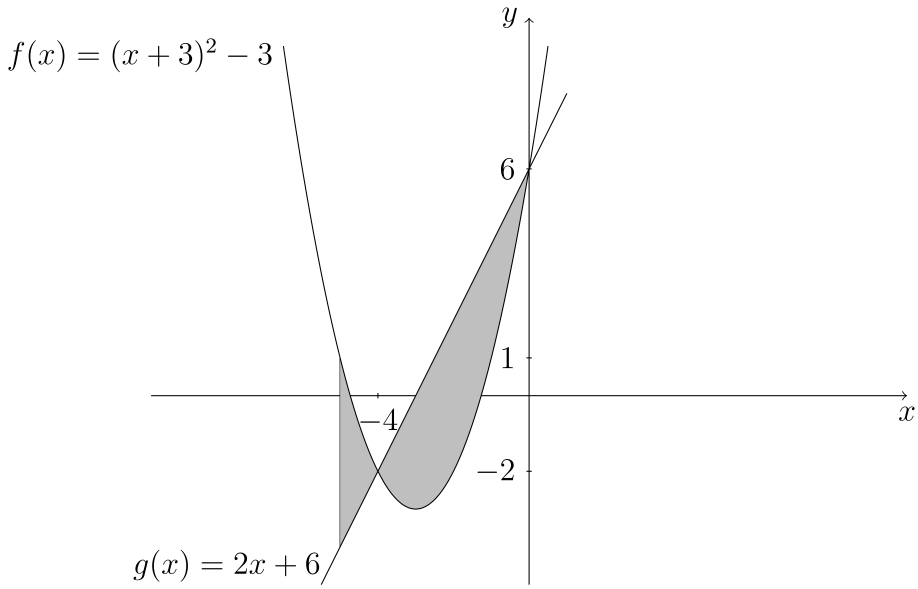

Figure 27

Examining the graph, we see that $f(x) \geq g(x)$ *over* $[-5; -4]$ *and* $g(x) \geq f(x)$ *over* $[-4; 0]$*. Thus, two integrals are needed to calculate the total area.*

$$S_1 = \int_{-5}^{-4} (f(x) - g(x))\,dx = \int_{-5}^{-4} \big(((x+3)^2 - 3) - (2x+6)\big)\,dx$$

$$= \int_{-5}^{-4} \big(x^2 + 6x + 9 - 3 - 2x - 6\big)\,dx = \int_{-5}^{-4} \big(x^2 + 4x\big)\,dx = \frac{x^3}{3} + 2x^2 \,\Big|_{-5}^{-4}$$

$$= \left(\frac{(-4)^3}{3} + 2(-4)^2\right) - \left(\frac{(-5)^3}{3} + 2(-5)^2\right) = -\frac{64}{3} + 32 + \frac{125}{3} - 50$$

$$= \frac{61}{3} - 18 = \frac{61}{3} - \frac{54}{3} = \frac{7}{3}.$$

$$S_2 = \int_{-4}^{0} (g(x) - f(x))\,dx = \int_{-4}^{0} \big((2x+6) - ((x+3)^2 - 3)\big)\,dx$$

$$= \int_{-4}^{0} \big(2x + 6 - x^2 - 6x - 9 + 3\big)\,dx = \int_{-4}^{0} \big(-x^2 - 4x\big)\,dx = -\frac{x^3}{3} - 2x^2 \,\Big|_{-4}^{0}$$

$$= 0 - \left(-\frac{(-4)^3}{3} - 2(-4)^2\right) = -\frac{64}{3} + 32 = -\frac{64}{3} + \frac{96}{3} = \frac{32}{3}.$$

The total area between the two graphs is

$$S = S_1 + S_2 = \frac{7}{3} + \frac{32}{3} = \frac{39}{3} = 13.$$

Applications

Income distribution

Economists use a Lorenz curve $y = L(x)$ to describe the distribution of income between families of a country. Any Lorenz curve is defined on the interval $[0, 1]$ with endpoints $(0; 0)$ and $(1; 1)$. It is continuous, increasing and concave upward. The points $(x; y)$ on this curve are determined by the cumulative percentage x of families at or below the given income level and the cumulative percentage y of families income received. Absolute equality of income distribution occurs when $a\%$ of families receive $a\%$ of income. In this case, the Lorenz curve equals $y = x$. The area between the Lorenz curve and the line $y = x$ shows how much the income distribution differs from absolute equality. The coefficient of inequality or Gini index is the ratio of the area S_1 between the Lorenz curve and $y = x$ to the area S_2 under $y = x$. The first area is equal $S_1 = \int\limits_0^1 (x - L(x))dx$ and the second (triangular) area S_2 is equal $S_2 = \frac{1}{2}$. Then

$$\text{Gini index} = 2\int\limits_0^1 (x - L(x))dx.$$

It is obvious that the Gini index is a number between 0 and 1. If the index is close to 0, the income is almost equally distributed. If the index is close to 1, the income is concentrated in a few families (Figure 28).

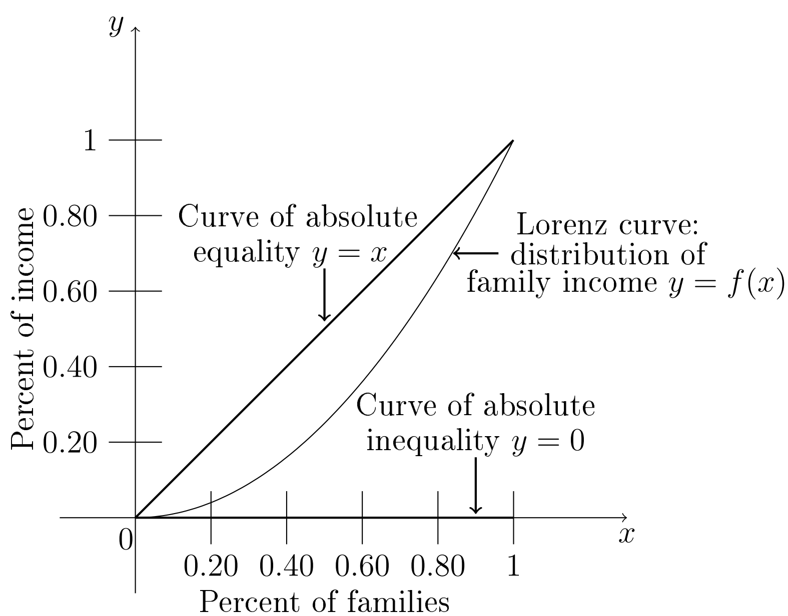

Figure 28

Example 4 *Suppose that the income distribution for a certain country is represented by the Lorenz curve $L(x) = \frac{5}{12}x^2 + \frac{7}{12}x$. Find the Gini index.*

Solution 4

$$\text{Gini index} = 2\int_0^1 \left(x - \left(\frac{5}{12}x^2 + \frac{7}{12}x\right)\right) dx = 2\int_0^1 \left(-\frac{5}{12}x^2 + \frac{5}{12}x\right) dx$$

$$= \frac{5}{6}\left(-\frac{x^3}{3} + \frac{x^2}{2}\right) \Big|_0^1 = \frac{5}{6}\left(-\frac{1^3}{3} + \frac{1^2}{2}\right) = \frac{5}{6} \cdot \frac{1}{6} = \frac{5}{36} \approx 0.14.$$

Consumers' and producers' surplus

As indicated earlier, the supply and demand of an item are usually related to its price. Suppose that $y = D(x)$ is a demand equation and $y = S(x)$ is a supply equation. Here, x stands for the quantity and y stands for the price. Suppose that p_0 is the current price and q_0 is the number of items that can be sold at that price. However, some consumers will be willing to pay more for an item than p_0. The total amount saved by all consumers who are willing to pay a price higher than p_0 is called the *consumers' surplus*. In Figure 29, the all region under the curve $y = D(x)$ is the total amount consumers are willing to spend for q_0 items. The area under the line $y = p_0$ illustrates the total amount consumers actually spend at the price p_0. The shaded area represents the consumers' surplus. Then

$$\text{Consumers' surplus} = \int_0^{q_0} (D(x) - p_0)dx.$$

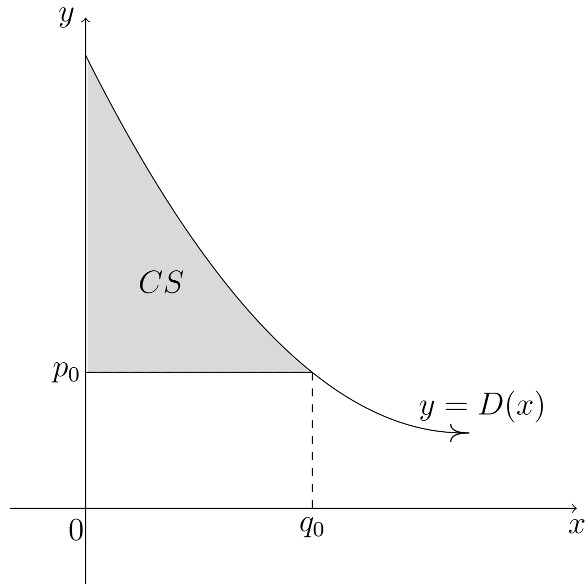

Figure 29

Similarly, if some producers will be wiling to supply a product at a price lower than p_0, the additional money that these producers gain from the higher price p_0 is called the producers' surplus. In Figure 30, all area under the line $y = p_0$ illustrates the total amount actually realized. The area under the curve $y = S(x)$ shows the minimum total amount the producers are willing to realize from the sale of q_0 items. The shaded area represents the producers' surplus. Then

$$\text{Producers' surplus} = \int_0^{q_0} (p_0 - S(x))dx.$$

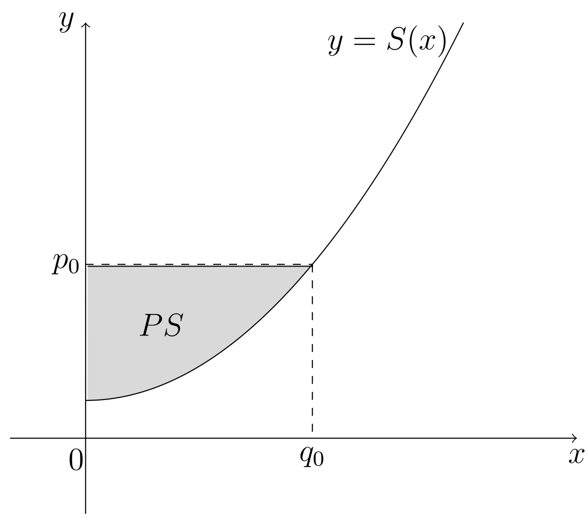

Figure 30

Example 5 *Find the consumers' surplus and producers' surplus at the equilibrium point level if* $D(x) = -x^2 - 25x + 500$ *and* $S(x) = x^2 + 5x$.

Solution 5 *Find the equilibrium point from the equation* $D(x) = S(x)$.

$$x^2 + 5x = -x^2 - 25x + 500$$

$$x^2 + 5x + x^2 + 25x - 500 = 0$$

$$2x^2 + 30x - 500 = 0$$

$$x^2 + 15x - 250 = 0$$

$$x = 10,\ -25.$$

Since x *cannot be negative, the equilibrium quantity is* $q_0 = 10$. *Moreover, the equilibrium price is* $p_0 = D(10) = S(10) = 150$.

Thus,

$$\text{Consumers' surplus} = \int_0^{10} (-x^2 - 25x + 500 - 150)dx = \int_0^{10} (-x^2 - 25x + 350)dx$$

$$= -\frac{x^3}{3} - \frac{25x^2}{2} + 350x \;|_0^{10} = -\frac{10^3}{3} - \frac{25 \cdot 10^2}{2} + 350 \cdot 10 = -\frac{1000}{3} - 25 \cdot 50 + 3500$$
$$= -\frac{1000}{3} + 2250 = \frac{-1000 + 6750}{3} = \frac{5750}{3} \approx \$1916.7.$$

$$\text{Producers' surplus} = \int_0^{10} (150 - x^2 - 5x)dx = 150x - \frac{x^3}{3} - \frac{5x^2}{2} \;|_0^{10}$$
$$= 150 \cdot 10 - \frac{10^3}{3} - \frac{5 \cdot 10^2}{2} = 1500 - \frac{1000}{3} - 250$$
$$= 1250 - \frac{1000}{3} = \frac{3750 - 1000}{3} = \frac{2750}{3} \approx \$916.7.$$

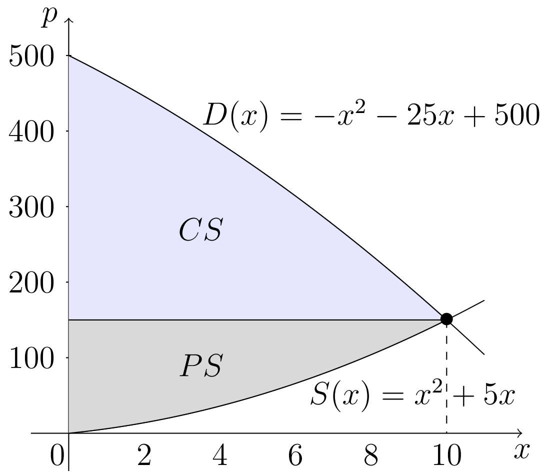

Figure 31