6.3 Definite integrals

Let f be a function defined on the interval $[a;b]$. We divide the interval $[a;b]$ into n subintervals of equal length $\Delta x = (b-a)/n$:

$$a = x_0 < x_1 < x_2 < ... < x_{i-1} < x_i < ... < x_{n-1} < x_n = b.$$

Then we choose arbitrary points c_i from each subintervals $[x_{i-1}; x_i]$ and find their values $f(c_i)$. Next we construct the sum:

$$S_n = f(c_1) \cdot \Delta x + f(c_2) \cdot \Delta x + ... + f(c_n) \cdot \Delta x = \sum_{i=1}^{n} f(c_i) \cdot \Delta x.$$

This sum is called *the Riemann integral sum.*

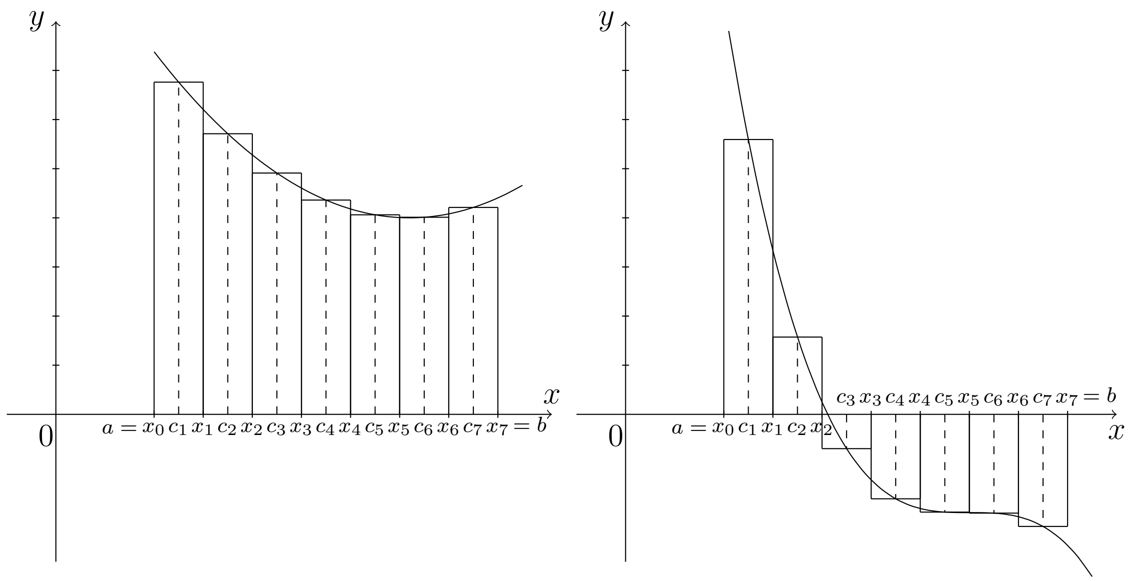

Figure 24

Theorem 1 *If f is continuous function on $[a;b]$, then there exists the limit of the Riemann integral sums when $n \to \infty$.*

Definition 1 *Let f be a continuous function on $[a;b]$. The limit of the Riemann integral sums when $n \to \infty$ is called the definite integral of f from a to b, denoted $\int_a^b f(x)dx$. Thus,*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \cdot \Delta x.$$

Here a and b are the lower and upper limits of integration, respectively.

Obviously, the definite integral has the following geometric interpretation: it represents the sum of the areas between the graph of f and the x-axis from a to b, where the areas above the x-axis are counted positively and the areas below x-axis are counted negatively (Figure 24).

Properties of definite integrals

1. $\int\limits_a^a f(x)dx = 0$;
2. $\int\limits_a^b kf(x)dx = k\int\limits_a^b f(x)dx$, where k is a constant;
3. $\int\limits_a^b f(x)dx = -\int\limits_b^a f(x)dx$;
4. $\int\limits_a^b (f(x) \pm g(x))\, dx = \int\limits_a^b f(x)dx \pm \int\limits_a^b g(x)dx$;
5. $\int\limits_a^b f(x)dx = \int\limits_a^c f(x)dx + \int\limits_c^b f(x)dx$, where $a < c < b$.

6.4 Fundamental theorem of calculus

Theorem 2 *If f is a continuous function on $[a;b]$, and $F(x)$ is an antiderivative of f, then*

$$\int\limits_a^b f(x)dx = F(x)\mid_a^b = F(b) - F(a).$$

Proof. Let us first prove the following fact. If the function $G(x)$ is defined on $[a;b]$ by the formula

$$G(x) = \int\limits_a^x f(t)dt,$$

then $G'(x) = f(x)$, i.e., $G(x)$ is an antiderivative of f.

Indeed, using the definition of derivative, we have

$$G'(x) = \lim_{\Delta x \to 0} \frac{G(x+\Delta x) - G(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\int\limits_a^{x+\Delta x} f(t)dt - \int\limits_a^x f(t)dt\right) =$$

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\int\limits_a^x f(t)dt + \int\limits_x^{x+\Delta x} f(t)dt - \int\limits_a^x f(t)dt\right) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\int\limits_x^{x+\Delta x} f(t)dt.$$

For some number c between x and $x+\Delta x$, by the definition of the definite integral we get

$$\int_{x}^{x+\Delta x} f(t)dt \approx \Delta x \cdot f(c).$$

Therefore,

$$G'(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \Delta x \cdot f(c) = \lim_{\Delta x \to 0} f(c).$$

Since $c \to x$ as $\Delta x \to 0$ and f is continuous, then

$$G'(x) = \lim_{c \to x} f(c) = f(x).$$

If $G(x)$ and $F(x)$ are both antiderivatives of f, then $G(x) = F(x) + C$. Hence,

$$\int_{a}^{x} f(t)dt = G(x) = F(x) + C.$$

Let $x = a$, then

$$\int_{a}^{a} f(t)dt = 0 = F(a) + C,$$

so $C = -F(a)$. Now let $x = b$, then

$$\int_{a}^{b} f(t)dt = F(b) + C = F(b) - F(a).$$

The proof is complete.

Example 1 *Evaluate the integral* $\int_{-1}^{3} x^2 dx$.

Solution 1

$$\int_{-1}^{3} x^2 dx = \frac{x^3}{3} \Big|_{-1}^{3} = \frac{3^3}{3} - \frac{(-1)^3}{3} = \frac{27}{3} + \frac{1}{3} = \frac{28}{3}.$$

6.5 Integration by substitution for definite integrals

We have already discussed the substitution technique for indefinite integral. The same formula can be used for definite integral, but here we also substitute the lower and upper bounds a and b by α and β, respectively, where $g(\alpha) = a$ and $g(\beta) = b$. Thus,

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(t)) \cdot g'(t)dt = F(g(t))|_\alpha^\beta = F(g(\beta)) - F(g(\alpha)) = F(b) - F(a).$$

Example 2 *Evaluate the integral* $\int_e^{e^2} \frac{dx}{x \ln x}$.

Solution 2 *Let* $t = \ln x$, *then* $dt = d(\ln x) = \frac{1}{x}dx$. *Moreover, from the old lower and upper bounds* $x = e$ *and* $x = e^2$, *we can respectively find new lower and upper bounds* $t = \ln e = 1$ *and* $t = \ln e^2 = 2$. *Replace the results into the substitution formula:*

$$\int_e^{e^2} \frac{dx}{x \ln x} = \int_e^{e^2} \frac{1}{\ln x} \cdot \frac{1}{x}dx = \int_1^2 \frac{1}{t}dt = \ln|t| \,|_1^2 = \ln 2 - \ln 1 = \ln 2.$$

6.6 Average value

We know the formula that allows to calculate the average of a finite number n of values a_1, a_2, ..., a_n that is

$$\text{Average} = \frac{a_1 + a_2 + ... + a_n}{n}.$$

How can we find the average of a continuous function over some interval $[a; b]$ with infinitely many values? Let us choose n numbers of values of f: $f(c_1)$, $f(c_2)$, ..., $f(c_n)$, where $a < c_1 < c_2 < ... < c_n < b$. Then their average can be calculated by

$$\text{Average} = \frac{f(c_1) + f(c_2) + ... + f(c_n)}{n}.$$

Multiply the formula by $\frac{b-a}{b-a}$

$$\text{Average} = \frac{b-a}{b-a} \cdot \frac{f(c_1) + f(c_2) + ... + f(c_n)}{n}$$
$$= \frac{1}{b-a} \cdot \frac{b-a}{n}(f(c_1) + f(c_2) + ... + f(c_n)).$$

We remember that $\frac{b-a}{n} = \Delta x$, then

$$\text{Average} = \frac{1}{b-a} \cdot \Delta x(f(c_1) + f(c_2) + ... + f(c_n))$$

$$= \frac{1}{b-a}\left(f(c_1)\Delta x + f(c_2)\Delta x + ... + f(c_n)\Delta x\right).$$

Infinitely many values over the interval means that $n \to \infty$. Thus,

$$(\text{Average of } f \text{ over } [a;b]) = \frac{1}{b-a}\lim_{n\to\infty}\left(f(c_1)\Delta x + f(c_2)\Delta x + ... + f(c_n)\Delta x\right)$$

$$= \frac{1}{b-a}\int_a^b f(x)dx.$$

Definition 2 *Average value of a continuous function f over $[a;b]$ is given by the formula*

$$(\textit{Average of } f \textit{ over } [a;b]) = \frac{1}{b-a}\int_a^b f(x)dx.$$

Example 3 *Find the average value of $f(x) = \frac{1}{x^2+x}$ over $[1;3]$.*

Solution 3

$$Average = \frac{1}{3-1}\int_1^3 \frac{1}{x^2+x}dx = \frac{1}{2}\int_1^3 \frac{1}{x(x+1)}dx = \frac{1}{2}\int_1^3 \frac{(x+1)-x}{x(x+1)}dx$$

$$= \frac{1}{2}\int_1^3 \left(\frac{(x+1)}{x(x+1)} - \frac{x}{x(x+1)}\right)dx = \frac{1}{2}\int_1^3 \left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= \frac{1}{2}\left(\int_1^3 \frac{dx}{x} - \int_1^3 \frac{dx}{x+1}\right) = \frac{1}{2}\left(\ln x \mid_1^3 - \ln(x+1) \mid_1^3\right)$$

$$= \frac{1}{2}\left(\ln 3 - \ln 1 - \ln 4 + \ln 2\right) = \frac{1}{2}\ln\frac{3\cdot 2}{4} = \frac{1}{2}\ln\frac{3}{2}.$$