5.3 Strategy for graphing function

Step 1. Find domain of f;

Step 2. Find intercepts of f;

Step 3. Find asymptotes of f;

Step 4. Find intervals of monotonicity and local extreme points of f;

Step 5. Find intervals of concavity and inflection points of f;

Step 6. Sketch graph of f.

Example 1 *Use the graphing strategy to analyze the function* $f(x) = \frac{x^3+4}{x^2}$ *and sketch its graph.*

Solution 1 *Step 1. Domain:* $D(f) = (-\infty; 0) \cup (0; +\infty)$.

Step 2. Intercepts: If $y = 0$, *then* $x^3 + 4 = 0$ *or* $x^3 = -4$, *so the* x*-intercept is* $x = -\sqrt[3]{4}$.

Since 0 *is not in the domain of* f, *there is no* y*-intercept.*

Step 3. Asymptotes: 1) The denominator is 0 *for* $x = 0$, *and*

$$\lim_{x\to 0+} \frac{x^3+4}{x^2} = +\infty,$$

$$\lim_{x\to 0-} \frac{x^3+4}{x^2} = +\infty.$$

Therefore, $x = 0$ *is a vertical asymptote.*

2) Since

$$\lim_{x\to +\infty} \frac{x^3+4}{x^2} = +\infty$$

and

$$\lim_{x\to -\infty} \frac{x^3+4}{x^2} = -\infty,$$

the function does not have horizontal asymptotes to the both directions.

3) Since

$$k = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{\frac{x^3+4}{x^2}}{x} = \lim_{x\to\pm\infty} \frac{x^3+4}{x^3} = 1$$

and

$$b = \lim_{x\to\pm\infty} (f(x) - kx) = \lim_{x\to\pm\infty} \left(\frac{x^3+4}{x^2} - x\right) = \lim_{x\to\pm\infty} \frac{x^3+4-x^3}{x^2}$$

$$= \lim_{x \to \pm\infty} \frac{4}{x^2} = 0,$$

$y = x$ is an oblique asymptote to the both directions.

Step 4. Intervals of monotonicity and local extreme points: 1) Derivative f':

$$f'(x) = \frac{(x^3+4)' \cdot x^2 - (x^2)' \cdot (x^3+4)}{x^4} = \frac{3x^2 \cdot x^2 - 2x \cdot (x^3+4)}{x^4}$$

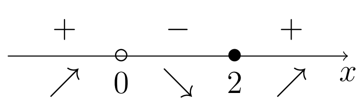
$$= \frac{3x^4 - 2x^4 - 8x}{x^4} = \frac{x^4 - 8x}{x^4} = \frac{x(x^3-8)}{x^4} = \frac{x^3-8}{x^3}.$$

2) Partition numbers for f':

$f' = 0$ if $x^3 - 8 = 0$ or $x^3 = 8$, then $x_1 = 2$ is a partition number;

f' does not exist if $x^3 = 0$, then $x_2 = 0$ is a partition number.

3) Sign chart for f':



Test numbers	
x	$f'(x)$
-1	$9 \quad (+)$
1	$-7\ (-)$
3	$\frac{19}{27} \quad (+)$

Conclusion: The sign chart indicates that f is increasing on $(-\infty; 0)$ and $(2; +\infty)$; f is decreasing on $(0; 2)$. Moreover, since 0 is not in the domain of f, it is not an extreme point. 2 is in the domain of f, so $f(2) = \frac{2^3+4}{2^2} = 3$ is a local minimum.

Step 5. Intervals of concavity and inflection points: 1) Second order derivative f'':

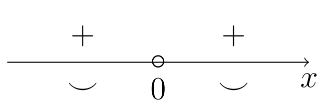
$f''(x) = \left(\frac{x^3-8}{x^3}\right)' = (1 - 8 \cdot x^{-3})' = 24x^{-4} = \frac{24}{x^4}$.

2) Partition numbers for f'':

$f'' \neq 0$ for any number of the domain of f'';

f'' does not exist if $x^4 = 0$, then 0 is a partition number.

3) Sign chart for f'':



Test numbers	
x	$f''(x)$
-1	$24 \quad (+)$
1	$24 \quad (+)$

Conclusion: The sign chart indicates that f is concave up on $(-\infty; 0)$ and $(0; +\infty)$. Moreover, there is no inflection point.

Step 6. Sketch graph of f (Figure 23).

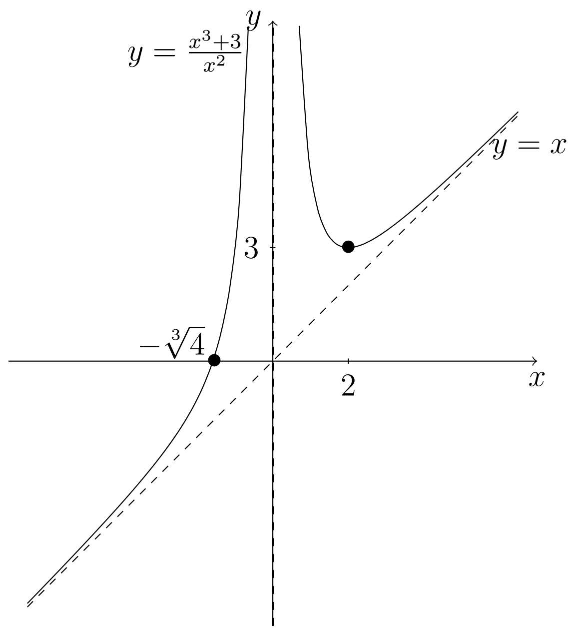


Figure 23