5 Derivatives and graphs

5.1 Intervals of monotonicity and local extreme points

Let us remind the following definition.

Definition 1 *If* $a < x_1 < x_2 < b$ *implies that* $f(x_1) < f(x_2)$*, then a function* $y = f(x)$ *is increasing on an interval* $(a; b)$*. If* $a < x_1 < x_2 < b$ *implies that* $f(x_1) > f(x_2)$*, then* f *is decreasing on* $(a; b)$*.*

Theorem 1 *Suppose that a function* $y = f(x)$ *is differentiable over an interval* $(a; b)$*.*

1. If $f'(x) > 0$ *for each* x *in the interval* $(a; b)$*, then* f *is increasing on* $(a; b)$*;*

2. if $f'(x) < 0$ *for each* x *in the interval* $(a; b)$*, then* f *is decreasing on* $(a; b)$*.*

Definition 2 $f(x_0)$ *is called a local maximum of* $y = f(x)$ *if* $f(x_0) > f(x_0 + h)$ *and* $f(x_0) > f(x_0 - h)$ *for any sufficiently small* h*;* $f(x_0)$ *is called a local minimum of* $f(x)$ *if* $f(x_0) < f(x_0 + h)$ *and* $f(x_0) < f(x_0 - h)$ *for any sufficiently small* h*.*

$f(x_0)$ *is a local extremum if it is either a local maximum or minimum.*

Theorem 2 *If* $y = f(x)$ *has a local extremum at* x_0*, then either* $f'(x_0) = 0$ *or* $f'(x_0)$ *does not exist.*

The inverse statement is not always correct. For example, let $f(x) = (x-1)^3 + 2$. Its derivative $f'(x) = 3(x-1)^2$. Then, $f'(1) = 0$. However, from the the graph of $f(x) = (x-1)^3 + 2$ (Figure 2) it is obvious that $x_0 = 1$ is not a local extreme point.

Definition 3 *The partition numbers for a function* $y = f(x)$ *are values of* x *such that* f *is discontinuous at* x *or* $f(x) = 0$*.*

Definition 4 *The partition number* x_0 *for* f' *in the domain of* f *is called the critical number;* $f(x_0)$ *and* $(x_0; f(x_0))$ *are called the critical value and critical point, respectively.*

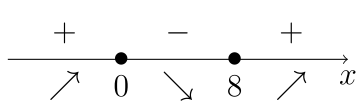
Remark 1 *From Definitions 3 and 4 it is obvious that f' may have partition numbers that are not critical if they are not in the domain of f.*

Theorem 3 *Suppose that $y = f(x)$ is differentiable over some neighborhood of a critical number x_0. If f' changes sign from positive to negative at x_0, then $f(x_0)$ is a local maximum; if f' changes sign from negative to positive at x_0, then $f(x_0)$ is a local minimum.*

The strategy for finding local extrema is the following: find partition numbers for f', construct a sign chart for f', locate the found partition numbers on the sign chart, select a test number in each obtained interval to determine the sign of f', indicate critical numbers among the partition numbers and draw a conclusion if they produce local maximum, local minimum or neither.

Example 1 *Find the intervals of monotonicity and local extreme points for $f(x) = \frac{x}{3} - \sqrt[3]{x^2}$.*

Solution 1 *Step 1. Domain:* $D(f) = (-\infty; +\infty)$.
Step 2. Derivative f': $f'(x) = \left(\frac{x}{3} - x^{\frac{2}{3}}\right)' = \frac{1}{3} - \frac{2}{3}x^{-\frac{1}{3}} = \frac{1}{3} - \frac{2}{3\sqrt[3]{x}} = \frac{\sqrt[3]{x}-2}{3\sqrt[3]{x}}$.
Step 3. Partition numbers for f':
1) $f' = 0$ if $\sqrt[3]{x} - 2 = 0$, then $x_1 = 8$ is a partition number;
2) f' does not exist if $3\sqrt[3]{x} = 0$, then $x_2 = 0$ is a partition number.
Step 4. Sign chart for f':



Test numbers	
x	$f'(x)$
-1	1 $(+)$
1	$-\frac{1}{3}$ $(-)$
27	$\frac{1}{9}$ $(+)$

Answer: The sign chart indicates that f is increasing on $(-\infty; 0)$ and $(8; +\infty)$; f is decreasing on $(0; 8)$. Moreover, since 0 and 8 are in the domain of f, they are also critical numbers. Thus, $f(0) = 0$ is a local maximum and $f(8) = \frac{8}{3} - \sqrt[3]{8} = \frac{8}{3} - 4 = -\frac{4}{3}$ is a local minimum.

Example 2 *Find the intervals of monotonicity and local extreme points for* $f(x) = \frac{1}{(x-2)^2}$.

Solution 2 *Step 1. Domain:* $D(f) = (-\infty; 2) \cup (2; +\infty)$.

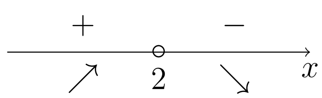
Step 2. Derivative f': $f'(x) = \left(\frac{1}{(x-2)^2}\right)' = \frac{-2}{(x-2)^3}$.

Step 3. Partition numbers for f':

1) $f' \neq 0$ *for any number of the domain of* f';

2) f' *does not exist if* $(x-2)^3 = 0$, *then* $x_0 = 2$ *is a partition number.*

Step 4. Sign chart for f':



Test numbers	
x	$f'(x)$
1	2 (+)
3	−2 (−)

Answer: The sign chart indicates that f *is increasing on* $(-\infty; 2)$ *and* f *is decreasing on* $(2; +\infty)$. *Moreover, since* 2 *is not in the domain of* f, *it is not a critical number. Thus,* f *has no extreme points.*

Second order derivative test

Sometimes, especially for polynomials, it is more convenient to use the test called the second order derivative test.

Theorem 4 *Let a function* $y = f(x)$ *be twice differentiable over some neighborhood of a number* x_0. *Suppose that* $f'(x_0) = 0$ *and* $f''(x_0) \neq 0$. *If* $f''(x_0) < 0$, *then* $f(x_0)$ *is a local maximum; if* $f''(x_0) > 0$, *then* $f(x_0)$ *is a local minimum.*

Example 3 *Find the local extreme points for* $f(x) = x^3 + 6x^2 - 63x + 7$.

Solution 3 *Step 1. Domain:* $D(f) = (-\infty; +\infty)$.

Step 2. Derivative f': $f'(x) = 3x^2 + 12x - 63 = 3(x^2 + 4x - 21) = 3(x-4)(x+7)$.

Step 3. Critical numbers for $f' = 0$:

$3(x-4)(x+7) = 0$, *then* $x_1 = 4$ *and* $x_2 = -7$ *are critical numbers.*

Step 4. Second order derivative f'': $f''(x) = 6x + 12$.

Step 5. Sign check for f'': $f''(4) = 24+12 = 36 > 0$, then $f(4) = 4^3+6\cdot 4^2-63\cdot 4+7 = 64+96-252+7 = 85$ is a local minimum.

$f''(-7) = -42+12 = -30 < 0$, then $f(-7) = (-7)^3+6\cdot(-7)^2-63\cdot(-7)+7 = -343+294+441+7 = 399$ is a local maximum.

Applications

Example 4 *A company produces and sells pencils. It has fixed costs (at 0 output) of \$4000 per month; and variable costs of \$1 per pencil. The price-demand equation is $P(x) = 6 - 0.001x$. What is the maximum profit?*

Solution 4 *The cost function is*

$$C(x) = 1\cdot x + 4000.$$

The revenue function is

$$R(x) = x\cdot(6 - 0.001x).$$

The profit function is

$$P(x) = R(x) - C(x) = x\cdot(6-0.001x) - x - 4000 = -0.001x^2 + 5x - 4000.$$

The marginal profit function is

$$P'(x) = (-0.001x^2 + 5x - 4000)' = -0.002x + 5.$$

Then

$$-0.002x + 5 = 0$$

$$x = 2500 \ \ \textit{critical number}$$

Since $P''(x) = -0.002 > 0$, by the second order derivative test the number $x = 2500$ maximizes the profit

$$P(2500) = -0.001\cdot 2500^2 + 5\cdot 2500 - 4000 = -6250 + 12500 - 4000 = \$\ 2250.$$

5.2 Intervals of concavity and inflection points

Definition 5 *We say that $y = f(x)$ is concave upward on an interval, if the graph of f lies above its tangent lines. We say that $y = f(x)$ is concave downward on an interval, if the graph of f lies below its tangent lines. The point of $y = f(x)$, where the graph of f changes concavity, is called the inflection point.*

Theorem 5 *Suppose that a function $y = f(x)$ is twice differentiable over an interval $(a; b)$.*
1. If $f''(x) > 0$ for each x in the interval $(a; b)$, then f is concave upward on $(a; b)$;
2. if $f''(x) < 0$ for each x in the interval $(a; b)$, then f is concave downward on $(a; b)$.

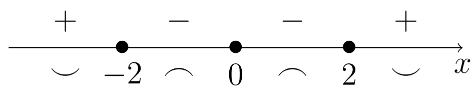
Theorem 6 *If $y = f(x)$ has an inflection point at x_0, then either $f''(x_0) = 0$ or $f''(x_0)$ does not exist.*

The inverse statement is not always correct. Thus, we need one more theorem.

Theorem 7 *Suppose that $y = f(x)$ is twice differentiable over some neighborhood of a number x_0, where x_0 is a partition number of f'' that belongs to the domain of f. If f'' changes sign at x_0, then $(x_0; f(x_0))$ is an inflection point.*

Example 5 *Find the intervals of concavity and inflection points for $f(x) = x^6 - 10x^4$.*

Solution 5 *Step 1. Domain: $D(f) = (-\infty; +\infty)$.*
Step 2. Derivative f': $f'(x) = 6x^5 - 40x^3$.
Step 3. Second order derivative f'': $f'' = 30x^4 - 120x^2 = 30x^2(x^2 - 4) = 30x^2(x - 2)(x + 2)$
Step 4. Partition numbers for f'':
1) $f'' = 0$ if $x^2(x - 2)(x + 2) = 0$, then $x_1 = -2$, $x_2 = 0$ and $x_3 = 2$ are partition numbers;
2) f'' exists for any real number.
Step 5. Sign chart for f'':



Test numbers	
x	$f''(x)$
-3	$+$
-1	$-$
1	$-$
3	$+$

Answer: The sign chart indicates that f is concave up on $(-\infty; -2)$ and $(2; +\infty)$; f is concave down on $(-2; 2)$. All three values 0, -2 and 2 are in the domain of f. However, since f'' does not change sign at 0, the function has not an inflection point at 0. Since f'' changes sign at -2 and 2, the function has inflection points at -2 and 2. Moreover, $f(-2) = (-2)^6 - 10 \cdot (-2)^4 = 64 - 160 = -96$ and $f(2) = -96$.