4.5 Equation of tangent line

We know that if a straight line passes through the point $M_1(x_1; y_1)$ and has the slope k, then its equation is given by the formula:

$$y - y_1 = k(x - x_1).$$

Moreover, the slope of a line tangent to the graph of $y = f(x)$ at the point $M_1(x_1; y_1)$ equals the derivative of this function at this point. Thus, $k = f'(x_1)$. Therefore, the equation of this tangent line is given by the formula:

$$y - y_1 = f'(x_1)(x - x_1).$$

Example 1 *Find the equation of the line tangent to the graph of* $f(x) = -(x+3)^2 - 5$ *at the point* $x_1 = -2$.

Solution 1 *Find* y_1*:*

$$y_1 = f(-2) = -(-2+3)^2 - 5 = -6.$$

Find $f'(x)$ *and the value* $f'(-2)$*:*

$$f'(x) = -2(x+3),$$

$$f'(-2) = -2(-2+3) = -2.$$

Find the equation of the tangent line:

$$y - (-6) = -2(x - (-2))$$

$$y = -2x - 10.$$

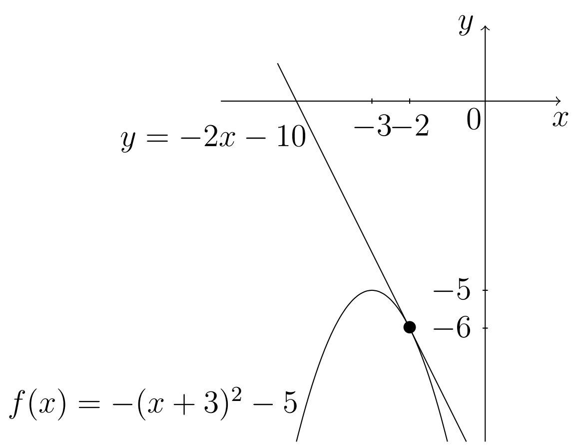

Figure 22

4.6 L'Hôpital's rules

The first L'Hôpital rule

Suppose the functions f and g are differentiable on the interval $(a;b)$ and $g'(x) \neq 0$ there. Let $a < c < b$. Suppose also that

1) $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to a} g(x) = 0$,
2) $\lim\limits_{x\to c} \frac{f'(x)}{g'(x)} = L$, where L is finite or $+\infty$ or ∞.

Then $\lim\limits_{x\to c} \frac{f(x)}{g(x)} = L$.

The second L'Hôpital rule

Suppose the functions f and g are differentiable on the interval $(a;b)$ and $g'(x) \neq 0$ there. Let $a < c < b$. Suppose also that

1) $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to a} g(x) = \pm\infty$,
2) $\lim\limits_{x\to c} \frac{f'(x)}{g'(x)} = L$, where L is finite or $+\infty$ or ∞.

Then $\lim\limits_{x\to c} \frac{f(x)}{g(x)} = L$.

Example 2 *Find the limit by L'Hôpital's rule* $\lim\limits_{x\to 1} \frac{\ln x}{x^2-1}$.

Solution 2 *It is obvious that* $(\ln x) \to 0$ *and* $(x^2 - 1) \to 0$ *when* $x \to 1$. *Thus, by L'Hôpital's rule we have*

$$\lim_{x\to 1} \frac{\ln x}{x^2 - 1} = \lim_{x\to 1} \frac{(\ln x)'}{(x^2 - 1)'} = \lim_{x\to 1} \frac{\frac{1}{x}}{2x} = \frac{\frac{1}{1}}{2 \cdot 1} = \frac{1}{2}.$$

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Applications

In business and economics, the rates of change are called not derivatives but *marginals*. So we can define *marginal cost*, *marginal revenue* and *marginal profit*.

Example 3 *The total cost (in dollars) of producing* x *units is given by*

$$C(x) = 0.01x^3 - 0.2x^2 + 10x + 2000.$$

1. Find the marginal cost function. 2. Find its value at a production level of 10 units and interpret the result.

Solution 3 *1.* $C'(x) = 0.03x^2 - 0.4x + 10.$

2. $C'(10) = 3 - 4 + 10 = \$\ 9.$

This means that if 10 units have been produced, then cost of producing the 11th unit is approximately \$9. Let us check.

Total cost of producing 11 units is

$$C(11) = 0.01 \cdot 11^3 - 0.2 \cdot 11^2 + 10 \cdot 11 + 2000 = 13.31 - 24.2 + 110 + 2000 = \$\ 2099.11$$

Total cost of producing 10 units is

$$C(10) = 0.01 \cdot 10^3 - 0.2 \cdot 10^2 + 10 \cdot 10 + 2000 = 10 - 20 + 100 + 2000 = \$\ 2090.$$

Exact cost of producing the 11th unit is

$$C(11) - C(10) = 2099.11 - 2090 = \$\ 9.11.$$