Equation of tangent line

We know that if a straight line passes through the point $M_1(x_1; y_1)$ and has the slope k, then its equation is given by the formula:

$$y - y_1 = k(x - x_1).$$

Moreover, the slope of a line tangent to the graph of $y = f(x)$ at the point $M_1(x_1; y_1)$ equals to the derivative of this function at this point. Thus, $k = f'(x_1)$. Therefore, the equation of this tangent line is given by the formula:

$$y - y_1 = f'(x_1)(x - x_1).$$

Example 1 *Find the equation of the line tangent to the graph of* $f(x) = -(x+3)^2-5$ *at the point* $x_1 = -2$.

Solution 1 *Find* y_1*:* $y_1 = f(-2) = -(-2+3)^2 - 5 = -6.$
Find $f'(x)$ *and the value* $f'(-2)$*:*

$$f'(x) = -2(x+3),$$

$$f'(-2) = -2(-2+3) = -2.$$

Find the equation of the tangent line:

$$y - (-6) = -2(x - (-2)) \quad \Rightarrow \quad y = -2x - 10.$$

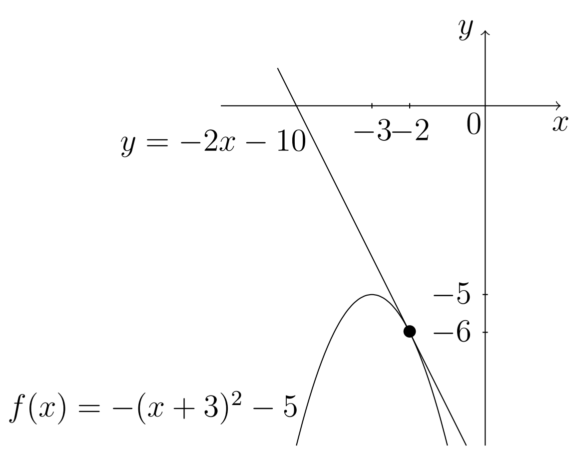

Figure 1

L'Hôpital's rules

The first L'Hôpital rule

Suppose the functions f and g are differentiable on the interval $(a; b)$ and $g'(x) \neq 0$ there. Let $a < c < b$. Suppose also that

1) $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to a} g(x) = 0$,
2) $\lim\limits_{x \to c} \frac{f'(x)}{g'(x)} = L$, where L is finite or $+\infty$ or ∞.

Then $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = L$.

The second L'Hôpital rule

Suppose the functions f and g are differentiable on the interval $(a; b)$ and $g'(x) \neq 0$ there. Let $a < c < b$. Suppose also that

1) $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to a} g(x) = \pm\infty$,
2) $\lim\limits_{x \to c} \frac{f'(x)}{g'(x)} = L$, where L is finite or $+\infty$ or ∞.

Then $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = L$.

Example 2 *Find the limit by L'Hôpital's rule* $\lim\limits_{x \to 1} \frac{\ln x}{x^2 - 1}$.

Solution 2 *It is obvious that* $(\ln x) \to 0$ *and* $(x^2 - 1) \to 0$ *when* $x \to 1$. *Thus, by L'Hôpital's rule we have*

$$\lim_{x \to 1} \frac{\ln x}{x^2 - 1} = \lim_{x \to 1} \frac{(\ln x)'}{(x^2 - 1)'} = \lim_{x \to 1} \frac{\frac{1}{x}}{2x} = \frac{\frac{1}{1}}{2 \cdot 1} = \frac{1}{2}.$$

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Applications

In business and economics, the rates of change are called not derivatives but *marginals*. So we can define *marginal cost*, *marginal revenue* and *marginal profit*.

Example 3 *The total cost (in dollars) of producing* x *units is given by*

$$C(x) = 0.01x^3 - 0.2x^2 + 10x + 2000.$$

1. Find the marginal cost function. 2. Find its value at a production level of 10 units and interpret the result.

Solution 3 *1.* $C'(x) = 0.03x^2 - 0.4x + 10.$

2. $C'(10) = 3 - 4 + 10 = \$\ 9.$

It means that if 10 units have been produced, then cost of producing the 11th unit is approximately \$9. Let us check.

Total cost of producing 11 units is

$$C(11) = 0.01 \cdot 11^3 - 0.2 \cdot 11^2 + 10 \cdot 11 + 2000 = 13.31 - 24.2 + 110 + 2000 = \$\ 2099.11$$

Total cost of producing 10 units is

$$C(10) = 0.01 \cdot 10^3 - 0.2 \cdot 10^2 + 10 \cdot 10 + 2000 = 10 - 20 + 100 + 2000 = \$\ 2090.$$

Exact cost of producing the 11th unit is

$$C(11) - C(10) = 2099.11 - 2090 = \$\ 9.11.$$

Asymptotes

An asymptote is a line such that the graph of a function approaches very close. Asymptotes are of three types: vertical, horizontal and oblique.

Definition 1 *The graph of $f(x)$ has a vertical asymptote $x = a$ if either*

$$\lim_{x \to a-} f(x) = \pm\infty$$

or

$$\lim_{x \to a+} f(x) = \pm\infty$$

or both.

How can we find vertical asymptotes? The domain will reveal vertical asymptotes.

Definition 2 *The graph of $f(x)$ has a horizontal asymptote $y = b$ if either*

$$\lim_{x \to -\infty} f(x) = b$$

or

$$\lim_{x \to +\infty} f(x) = b$$

or both.

Definition 3 *The graph of $f(x)$ has an oblique asymptote $y = kx + b$ if either*

$$\lim_{x \to -\infty} (f(x) - (kx + b)) = 0$$

or

$$\lim_{x \to +\infty} (f(x) - (kx + b)) = 0$$

or both.

Remark 1 *It is obvious that a horizontal asymptote is a partial case of an oblique asymptote when $k = 0$. Therefore, the oblique asymptotes will only be found when there are no horizontal asymptotes.*

How can we find oblique asymptotes? Let

$$\lim_{x \to +\infty} (f(x) - (kx + b)) = 0.$$

It means that

$$f(x) - (kx + b) = \alpha, \quad \text{where} \quad \alpha \to 0 \quad \text{when} \quad x \to +\infty.$$

Then

$$\begin{aligned} f(x) &= kx + b + \alpha \\ \frac{f(x)}{x} &= k + \frac{b}{x} + \frac{\alpha}{x} \\ \lim_{x \to +\infty} \frac{f(x)}{x} &= \lim_{x \to +\infty} \left(k + \frac{b}{x} + \frac{\alpha}{x} \right). \end{aligned}$$

Therefore,

$$k = \lim_{x \to +\infty} \frac{f(x)}{x}$$

and respectively,

$$b = \lim_{x \to +\infty} (f(x) - kx).$$

Similarly,

$$k = \lim_{x \to -\infty} \frac{f(x)}{x}$$

and respectively,

$$b = \lim_{x \to -\infty} (f(x) - kx).$$

Example 4 *Find the asymptotes for* $f(x) = \frac{x^2+1}{4x-1}$.

Solution 4 *1. The vertical asymptotes are found from the zeroes of the denominator:*

$$4x - 1 = 0$$

$$x = \frac{1}{4}.$$

Indeed,

$$\lim_{x \to \frac{1}{4}-} \frac{x^2+1}{4x-1} = -\infty$$

and

$$\lim_{x \to \frac{1}{4}+} \frac{x^2+1}{4x-1} = +\infty.$$

Therefore, $x = \frac{1}{4}$ *is a vertical asymptote.*

2. By the rule for limits at infinity for rational functions, we have

$$\lim_{x \to +\infty} \frac{x^2+1}{4x-1} = +\infty$$

and

$$\lim_{x \to -\infty} \frac{x^2+1}{4x-1} = -\infty.$$

Therefore, the function does not have horizontal asymptotes to the both directions.

3. By the rule for limits at infinity for rational functions, we have

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{\frac{x^2+1}{4x-1}}{x} = \lim_{x \to \pm\infty} \frac{x^2+1}{4x^2-x} = \frac{1}{4}$$

and

$$b = \lim_{x \to \pm\infty} (f(x) - kx) = \lim_{x \to \pm\infty} \left(\frac{x^2+1}{4x-1} - \frac{1}{4} \cdot x \right) = \lim_{x \to \pm\infty} \frac{4x^2+4-4x^2+x}{(4x-1)4}$$

$$= \lim_{x \to \pm\infty} \frac{4+x}{16x-4} = \frac{1}{16}.$$

Therefore, $y = \frac{1}{4}x + \frac{1}{16}$ *is an oblique asymptote to the both directions.*

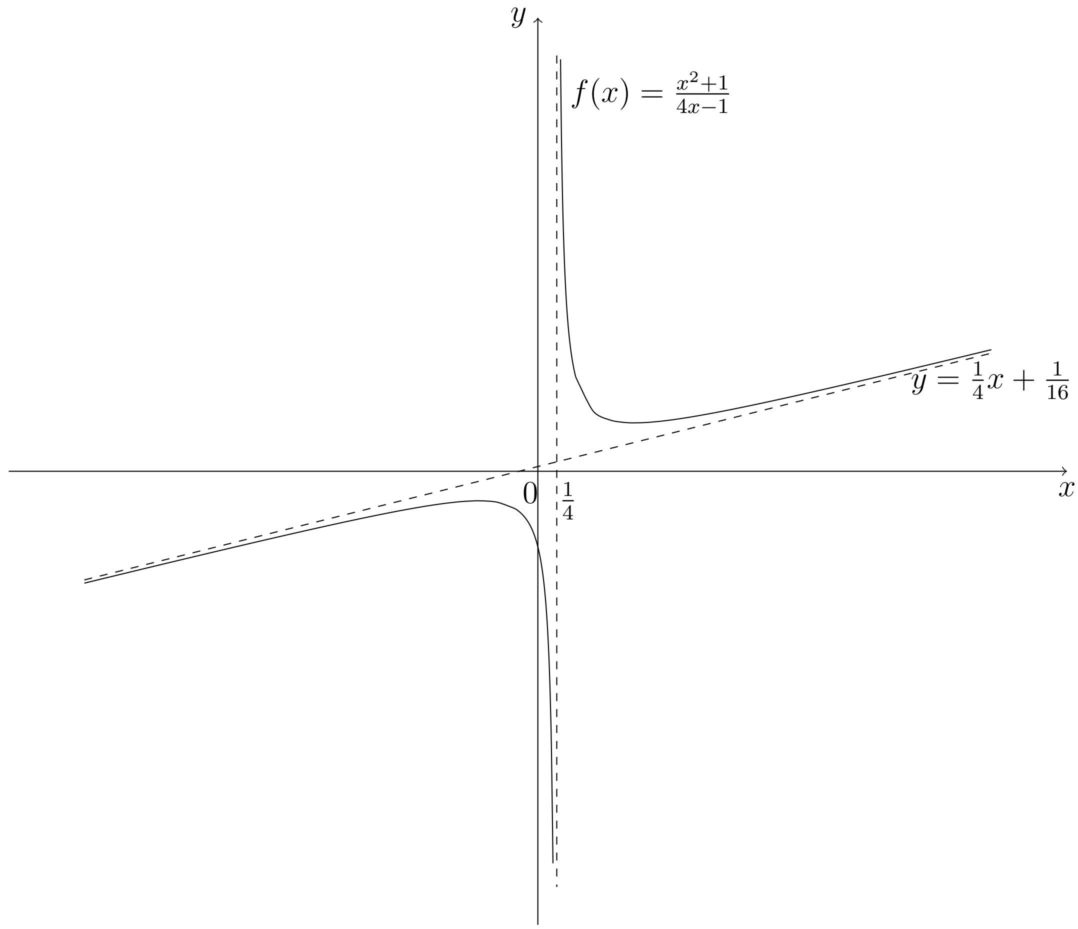

Figure 2