3 Limits of functions

3.1 Definitions

The idea of "limit" is to examine the behavior of a function $y = f(x)$ near some value $x = a$, but not at $x = a$. Let us consider two examples.

Example 1 *What happens to the values of $f(x) = 2x$ when x is very close to $x = 3$?*

Solution 1 *The answer is obvious from the following two tables:*

x	2	2.5	2.9	2.99	2.999	2.9999	...
$f(x)$	4	5	5.8	5.98	5.998	5.9998	...

x	4	3.5	3.1	3.01	3.001	3.0001	...
$f(x)$	8	7	6.2	6.02	6.002	6.0002	...

In the first table, x approaches 3 from the left. In the second table, x approaches 3 from the right. The tables suggest that as x gets closer and closer to 3 from both directions, the corresponding value of $f(x)$ gets closer and closer to 6.

This fact can be written as

$$\lim_{x \to 3-} 2x = 6$$

and

$$\lim_{x \to 3+} 2x = 6$$

for the left and right approaches, respectively. If answers for both sides are equal, we say that limit exists and write:

$$\lim_{x \to 3} 2x = 6.$$

Example 2 *What happens to the values of $f(x) = \frac{x}{|x|}$ when x is very close to $x = 0$?*

Solution 2 *Again the answer comes from the following two tables:*

x	-1	-0.5	-0.1	-0.01	-0.001	-0.0001	...
$f(x)$	-1	-1	-1	-1	-1	-1	...

x	*1*	*0.5*	*0.1*	*0.01*	*0.001*	*0.0001*	...
$f(x)$	*1*	*1*	*1*	*1*	*1*	*1*	...

The first table suggests that as x gets closer and closer to 0 from the left, the corresponding value of $f(x)$ equals -1. The second table suggests that as x gets closer and closer to 0 from the right, the corresponding value of $f(x)$ equals 1. This means that

$$\lim_{x \to 0-} \frac{x}{|x|} = -1$$

and

$$\lim_{x \to 0+} \frac{x}{|x|} = 1.$$

The answers are different, so we say that limit does not exist.

Based on these examples, we can write the following three informal definitions and theorem.

Definition 1 *We write*

$$\lim_{x \to a} f(x) = A$$

if the functional value $f(x)$ is close to the real number A whenever x is close to, but not equal to, a (on both sides of a).

Definition 2 *We write*

$$\lim_{x \to a-} f(x) = B$$

and call B the limit from the left or the left-hand limit if $f(x)$ is close to B whenever x is close to, but to the left of a ($x < a$).

Definition 3 *We write*

$$\lim_{x \to a+} f(x) = C$$

and call C the limit from the right or the right-hand limit if $f(x)$ is close to C whenever x is close to, but to the right of a ($x > a$).

Theorem 1 *For a limit to exist, the limit from the left and limit from the right must exist and be equal. That is*

$$\lim_{x \to a} f(x) = A \quad \text{if and only if} \quad \lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = A.$$

Properties of limits

Let f and g be two functions such that

$$\lim_{x \to a} f(x) = A \quad \text{and} \quad \lim_{x \to a} g(x) = B,$$

where A and B are real numbers (both limits exist). Then:

1. $\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = A \pm B$;
2. $\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = A \cdot B$;
3. $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x) = kA$ for any constant k;
4. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{A}{B}$;
5. $\lim_{x \to a} (f(x))^r = \left(\lim_{x \to a} f(x)\right)^r = A^r$ for any real number r such that A^r exists.

3.2 Infinite limits

Suppose that $\lim_{x \to a} f(x) = A$, where A is not a finite real number but "infinity". This means that "positive infinity $+\infty$" and "negative infinity $-\infty$" do not denote numbers. They are just convenient notations to express that $f(x)$ takes very large or very small values. To illustrate this fact, let us again consider an example.

Example 3 *What happens to the values of $f(x) = \frac{1}{3-x}$ when x is very close to $x = 3$?*

Solution 3 *The answer is based on the following two tables:*

x	*2.9*	*2.99*	*2.999*	...	*2.999999*	...	*2.999999999*	...
$f(x)$	*10*	*100*	*1000*	...	*1000000*	...	*1000000000*	...

x	*3.1*	*3.01*	*3.001*	...	*3.000001*	...	*3.000000001*	...
$f(x)$	*-10*	*-100*	*-1000*	...	*-1000000*	...	*-1000000000*	...

The first table shows that as x approaches 3 from the left, the corresponding values of $f(x)$ get very large. The second table shows that as x approaches 3 from the right, the corresponding values of $f(x)$ get very small. So, we write

$$\lim_{x \to 3-} \frac{1}{3-x} = +\infty$$

and

$$\lim_{x \to 3+} \frac{1}{3-x} = -\infty,$$

and say that the limit from the left of $f(x)$ is infinity and the limit from the right of $f(x)$ is negative infinity.

This means that sometimes either on the left side or on the right side or on both sides of the specified point $x = a$ the values of f infinitely increase or/and infinitely decrease. For example,

1. values of $f(x)$ increase without bound on both sides of a (Figure 16, A):

$$\lim_{x \to a} f(x) = +\infty,$$

2. values of $f(x)$ decrease without bound on both sides of a (Figure 16, B):

$$\lim_{x \to a} f(x) = -\infty,$$

3. values of $f(x)$ increase without bound on the left of a and decrease on the right of a (Figure 16, C):

$$\lim_{x \to a-} f(x) = +\infty \quad \text{and} \quad \lim_{x \to a+} f(x) = -\infty,$$

4. values of $f(x)$ decrease without bound on the left of a and increase on the right of a (Figure 16, D):

$$\lim_{x \to a-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to a+} f(x) = +\infty.$$

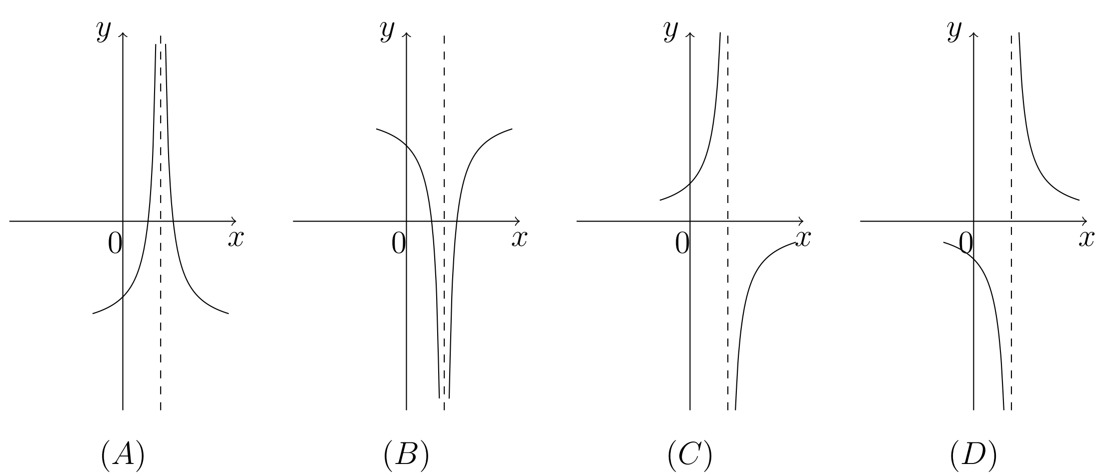

Figure 16

In all the situations just described we do not say that $\lim_{x\to a} f(x)$ exists. Rather, we say that limit does not exist because $f(x)$ becomes very large or very small near $x = a$.

3.3 Limits at infinity

Now we consider the behavior of a function $f(x)$ when x is very large or very small. This means that x tends to "positive infinity $+\infty$" or "negative infinity $-\infty$" but not to a number.

Example 4 *Describe the behavior of $f(x) = e^x$ when x is very large and very small.*

Solution 4 *The behavior of $f(x) = e^x$ can be described by its graph (Figure 13, A).*

When x is very large, the corresponding values of $f(x)$ get also very large. Thus, we write

$$\lim_{x\to+\infty} e^x = +\infty.$$

The left side of the graph appears to coincide with the x-axis. Indeed, when x gets smaller and smaller, the corresponding values are very close to 0. So, we write

$$\lim_{x\to-\infty} e^x = 0.$$

Example 5 *Describe the behavior of $f(x) = \frac{3}{1+e^x} + 5$ when x is very large and very small.*

Solution 5 *It is obvious that if we divide a finite real number by a very large number, we get a number close to zero. From Example 4 we know that e^x gets very large whenever x is very large. Therefore, the first term $\frac{3}{1+e^x}$ tends to zero when x tends to $+\infty$. Thus, if we replace the first term by zero, we get*

$$\lim_{x\to+\infty} \left(\frac{3}{1+e^x} + 5\right) = 0 + 5 = 5.$$

From Example 4 we know that if x is close to a very small number, then e^x is close to 0*. If we substitute* 0 *instead of e^x, we get*

$$\lim_{x\to-\infty}\left(\frac{3}{1+e^x}+5\right)=\frac{3}{1+0}+5=3+5=8.$$

These answers can be explained by the graph of $f(x)=\frac{3}{1+e^x}+5$ (Figure 17). The right side of the graph is very close to the line $y=5$ when x tends to $+\infty$. The left side of the graph appears to coincide with the line $y=8$ when x tends to $-\infty$.

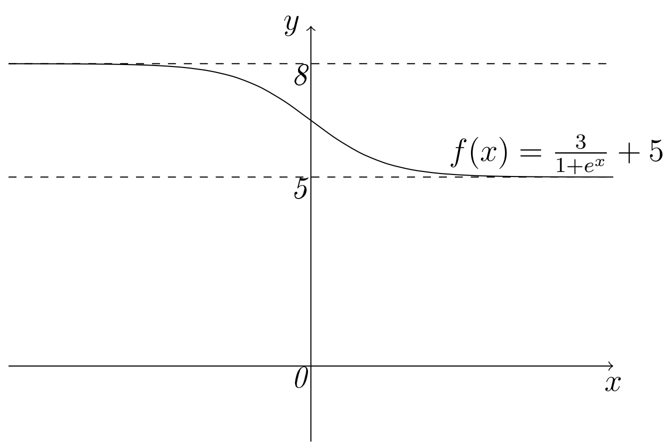

Figure 17

Examples 4 and 5 show that, in general, the behaviors of functions in two opposite directions are different. However, sometimes two answers can be equal.

Example 6 *Describe the behavior of $f(x)=\frac{1}{x}$ when x is very large and very small.*

Solution 6 *Make a table of values for very large x:*

x	*1000*	...	*1000000*	...	*1000000000*	...
$f(x)$	*0.001*	...	*0.000001*	...	*0.000000001*	...

The table suggests that

$$\lim_{x\to+\infty}\frac{1}{x}=0.$$

Make a table of values for very small x:

x	*-1000*	...	*-1000000*	...	*-1000000000*	...
$f(x)$	*-0.001*	...	*-0.000001*	...	*-0.000000001*	...

The table suggests that

$$\lim_{x\to-\infty}\frac{1}{x}=0.$$

Since the answers coincide, we could combine them and write

$$\lim_{x\to\pm\infty}\frac{1}{x}=0.$$

The solutions of the next three examples are based on Example 6.

Example 7 *Find the limit* $\lim_{x\to\pm\infty}\frac{5x^2-3x+1}{3x^2+2x+2}$.

Solution 7

$$\lim_{x\to\pm\infty}\frac{5x^2-3x+1}{3x^2+2x+2}=\lim_{x\to\pm\infty}\frac{x^2(5-\frac{3}{x}+\frac{1}{x^2})}{x^2(3+\frac{2}{x}+\frac{2}{x^2})}=\lim_{x\to\pm\infty}\frac{5-3\cdot\frac{1}{x}+\frac{1}{x}\cdot\frac{1}{x}}{3+2\cdot\frac{1}{x}+2\cdot\frac{1}{x}\cdot\frac{1}{x}}$$
$$=\frac{5-3\cdot 0+0\cdot 0}{3+2\cdot 0+2\cdot 0\cdot 0}=\frac{5}{3}.$$

Example 8 *Find the limit* $\lim_{x\to\pm\infty}\frac{5x^2-3x+1}{3x^3+2x+2}$.

Solution 8

$$\lim_{x\to\pm\infty}\frac{5x^2-3x+1}{3x^3+2x+2}=\lim_{x\to\pm\infty}\frac{x^2(5-\frac{3}{x}+\frac{1}{x^2})}{x^3(3+\frac{2}{x^2}+\frac{2}{x^3})}=\lim_{x\to\pm\infty}\frac{1}{x}\cdot\frac{5-3\cdot\frac{1}{x}+\frac{1}{x}\cdot\frac{1}{x}}{3+2\cdot\frac{1}{x}\cdot\frac{1}{x}+2\cdot\frac{1}{x}\cdot\frac{1}{x}\cdot\frac{1}{x}}$$
$$=0\cdot\frac{5-3\cdot 0+0\cdot 0}{3+2\cdot 0\cdot 0+2\cdot 0\cdot 0\cdot 0}=0\cdot\frac{5}{3}=0.$$

Example 9 *Find the limit* $\lim_{x\to\pm\infty}\frac{5x^3-3x+1}{3x^2+2x+2}$.

Solution 9

$$\lim_{x\to\pm\infty}\frac{5x^3-3x+1}{3x^2+2x+2}=\lim_{x\to\pm\infty}\frac{x^3(5-\frac{3}{x^2}+\frac{1}{x^3})}{x^2(3+\frac{2}{x}+\frac{2}{x^2})}=\lim_{x\to\pm\infty}x\cdot\frac{5-3\cdot\frac{1}{x}\cdot\frac{1}{x}+\frac{1}{x}\cdot\frac{1}{x}\cdot\frac{1}{x}}{3+2\cdot\frac{1}{x}+2\cdot\frac{1}{x}\cdot\frac{1}{x}}$$
$$=\frac{5}{3}\lim_{x\to\pm\infty}x=\pm\infty.$$

The technique in Examples 7, 8 and 9 carries over any rational function $f(x)=\frac{n(x)}{d(x)}$. Thus, we can write some general rule for *limits at infinity for rational functions.*

Suppose that

$$n(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$$

$$d(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0,$$

then

$$\lim_{x \to \pm\infty} \frac{n(x)}{d(x)} = \frac{a_n}{b_m} \text{ if } n = m;$$

$$\lim_{x \to \pm\infty} \frac{n(x)}{d(x)} = 0 \text{ if } n < m;$$

$$\lim_{x \to \pm\infty} \frac{n(x)}{d(x)} \text{ does not exist if } n > m.$$

3.4 Indeterminate forms

In most situations discussed above, limits $\lim_{x \to a} f(x)$ are performed by replacing x by a in $f(x)$. Sometimes, the expression obtained after this substitution does not give enough information to determine the limit and it is known as an *indeterminate form*. For example, if $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$ is said to be a $\frac{0}{0}$ indeterminate form.

The most common indeterminate forms are $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 1^∞, 0^0, ∞^0.

Example 10 *Find the limit* $\lim_{x \to 5} \frac{x-5}{x^2-25}$.

Solution 10 *When we replace x by 5 in the expression $\frac{x-5}{x^2-25}$ we get the $\frac{0}{0}$ indeterminate form. Since x is close to, but not equal to 5, we have*

$$\frac{x-5}{x^2-25} = \frac{x-5}{(x-5)(x+5)} = \frac{1}{x+5}.$$

Finally, we obtain

$$\lim_{x \to 5} \frac{x-5}{x^2-25} = \lim_{x \to 5} \frac{1}{x+5} = \frac{1}{5+5} = \frac{1}{10}.$$

3.5 Fundamental limits

There is a class of limits called the fundamental limits. Let us present two of them.

1.
$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e.$$

This limit has been already discussed in Section "Base e exponential function". If we replace x by $\frac{1}{t}$, the limit can be rewritten in the form:

$$\lim_{t\to 0}(1+t)^{\frac{1}{t}} = e.$$

Example 11 *Find the limit* $\lim_{x\to\infty}\left(1+\frac{7}{2x-4}\right)^{\frac{x}{3}}$.

Solution 11 *When we consider x to be very large or very small, the expression $\left(1+\frac{7}{2x-4}\right)^{\frac{x}{3}}$ has the 1^{∞} indeterminate form. This limit can be evaluated by the introduction of a new variable t such that $t=\frac{2x-4}{7}$. Hence, $x=\frac{7}{2}t+2$. Moreover, if $x\to\infty$, then $t\to\infty$. So, we have*

$$\lim_{x\to\infty}\left(1+\frac{7}{2x-4}\right)^{\frac{x}{3}} = \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{\frac{\frac{7}{2}t+2}{3}} = \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{\frac{7}{6}t+\frac{2}{3}}$$

$$= \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{\frac{7}{6}t}\left(1+\frac{1}{t}\right)^{\frac{2}{3}} = \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{\frac{7}{6}t}\cdot\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{\frac{2}{3}}$$

$$= \lim_{t\to\infty}\left(\left(1+\frac{1}{t}\right)^{t}\right)^{\frac{7}{6}}\cdot 1 = e^{\frac{7}{6}}.$$

2.

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

The graphs of $y=\sin x$ and $y=x$ (Figure 18) geometrically explain the validity of this formula. It is obvious that at the neighborhood of 0 the values of these two functions are very close, so their ratio is approximately equal to 1.

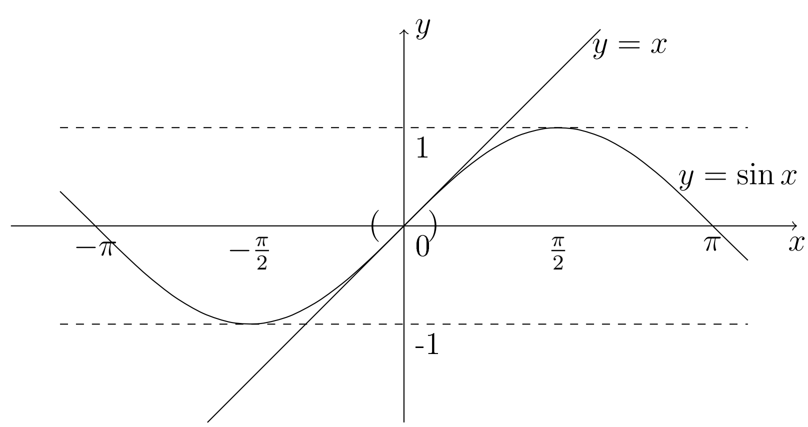

Figure 18

Example 12 *Find the limit* $\lim_{x \to 0} \frac{\sin 5x}{x}$.

Solution 12 *When we replace* x *by* 0 *in the expression* $\frac{\sin 5x}{x}$ *we get the* $\frac{0}{0}$ *indeterminate form. This limit can be evaluated by the introduction of a new variable* t *such that* $t = 5x$. *Hence,* $x = \frac{t}{5}$. *Moreover, if* $x \to 0$, *then* $t \to 0$. *So, we have*

$$\lim_{x \to 0} \frac{\sin 5x}{x} = \lim_{t \to 0} \frac{\sin t}{\frac{t}{5}} = 5 \lim_{t \to 0} \frac{\sin t}{t} = 5 \cdot 1 = 5.$$

Applications

Suppose that compounding periods m gets larger and larger in the general compound interest formula:

$$A = P\left(1 + \frac{r}{m}\right)^{mt}.$$

This equivalently means that

$$A = \lim_{m \to +\infty} P\left(1 + \frac{r}{m}\right)^{mt}$$

or

$$A = P \cdot \lim_{m \to +\infty} \left(\left(1 + \frac{r}{m}\right)^{\frac{m}{r}}\right)^{rt} = Pe^{rt}.$$

Thus, if a principal P is invested at an annual rate r compounded continuously, then the amount A in the account at the end of t years is given by

$$A = Pe^{rt}.$$

Example 13 *Suppose $5000 is invested in an account paying 9% compounded continuously. How much money will be in the account after 5 years?*

Solution 13

$$A = Pe^{rt} = 5000 \cdot e^{0.09 \cdot 5} = 5000 \cdot e^{0.45} \approx 5000 \cdot 1.568239 = \$\ 7841.20.$$

3.6 Continuity

Limits are used to define the continuity of a function at a point.

Definition 4 *A function f is continuous at $x = c$ if*

1. $f(c)$ is defined;

2. $\lim_{x \to c} f(x)$ exists;

3. $\lim_{x \to c} f(x) = f(c)$.

If at least one of the conditions in the definition fails, then f is discontinuous at $x = c$.

Definition 5 *A function f is continuous on the interval $(a; b)$ if it is continuous at each point on $(a; b)$.*

Example 14 *Are the functions $f(x) = 3x^2 - 2x + 5$ and $g(x) = \frac{x}{|x|}$ continuous at $x = 0$?*

Solution 14 *1. $f(0) = 5$ is defined. Moreover, $\lim_{x \to 0}(3x^2 - 2x + 5) = 5$. This means that $\lim_{x \to 0} f(x) = f(0)$. Therefore, f is continuous at 0.*
2. $g(0) = 0$ is defined. However, $\lim_{x \to 0} g(x)$ does not exist (see Example 2). Therefore, g is discontinuous at 0.

3.7 Asymptotes

An asymptote is a line such that the graph of a function approaches it very closely. Asymptotes are of three types: vertical, horizontal and oblique.

Definition 6 *The graph of $f(x)$ has a vertical asymptote $x = a$ if either*

$$\lim_{x \to a-} f(x) = \pm\infty$$

or

$$\lim_{x \to a+} f(x) = \pm\infty$$

or both.

How can we find vertical asymptotes? The domain determines the vertical asymptotes.

Definition 7 *The graph of $f(x)$ has a horizontal asymptote $y = b$ if either*

$$\lim_{x \to -\infty} f(x) = b$$

or

$$\lim_{x \to +\infty} f(x) = b$$

or both.

Definition 8 *The graph of $f(x)$ has an oblique asymptote $y = kx + b$ if either*

$$\lim_{x \to -\infty} (f(x) - (kx + b)) = 0$$

or

$$\lim_{x \to +\infty} (f(x) - (kx + b)) = 0$$

or both.

Remark 1 *It is obvious that a horizontal asymptote is a partial case of an oblique asymptote when $k = 0$. Therefore, the oblique asymptotes will only be found when there are no horizontal asymptotes.*

How can we find oblique asymptotes? Let

$$\lim_{x \to +\infty} (f(x) - (kx + b)) = 0.$$

This means that

$$f(x) - (kx + b) = \alpha, \quad \text{where} \quad \alpha \to 0 \quad \text{when} \quad x \to +\infty.$$

Then

$$f(x) = kx + b + \alpha$$
$$\frac{f(x)}{x} = k + \frac{b}{x} + \frac{\alpha}{x}$$
$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \left(k + \frac{b}{x} + \frac{\alpha}{x} \right).$$

Therefore,

$$k = \lim_{x \to +\infty} \frac{f(x)}{x}$$

and respectively,

$$b = \lim_{x \to +\infty} \left(f(x) - kx \right).$$

Similarly,

$$k = \lim_{x \to -\infty} \frac{f(x)}{x}$$

and respectively,

$$b = \lim_{x \to -\infty} \left(f(x) - kx \right).$$

Example 15 *Find the asymptotes for* $f(x) = \frac{x^2+1}{4x-1}$.

Solution 15 *1. The vertical asymptotes are found from the zeroes of the denominator:*

$$4x - 1 = 0$$

$$x = \frac{1}{4}.$$

Indeed,

$$\lim_{x \to \frac{1}{4}-} \frac{x^2+1}{4x-1} = -\infty$$

and

$$\lim_{x \to \frac{1}{4}+} \frac{x^2+1}{4x-1} = +\infty.$$

Therefore, $x = \frac{1}{4}$ *is a vertical asymptote.*

2. By the rule for limits at infinity for rational functions, we have

$$\lim_{x \to +\infty} \frac{x^2+1}{4x-1} = +\infty$$

and

$$\lim_{x \to -\infty} \frac{x^2+1}{4x-1} = -\infty.$$

Therefore, the function does not have horizontal asymptotes in both directions.

3. By the rule for limits at infinity for rational functions, we have

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{\frac{x^2+1}{4x-1}}{x} = \lim_{x \to \pm\infty} \frac{x^2+1}{4x^2-x} = \frac{1}{4}$$

and

$$b = \lim_{x \to \pm\infty} \left(f(x) - kx \right) = \lim_{x \to \pm\infty} \left(\frac{x^2+1}{4x-1} - \frac{1}{4} \cdot x \right) = \lim_{x \to \pm\infty} \frac{4x^2+4-4x^2+x}{(4x-1)4}$$

$$= \lim_{x \to \pm\infty} \frac{4+x}{16x-4} = \frac{1}{16}.$$

Therefore, $y = \frac{1}{4}x + \frac{1}{16}$ *is an oblique asymptote in both directions.*

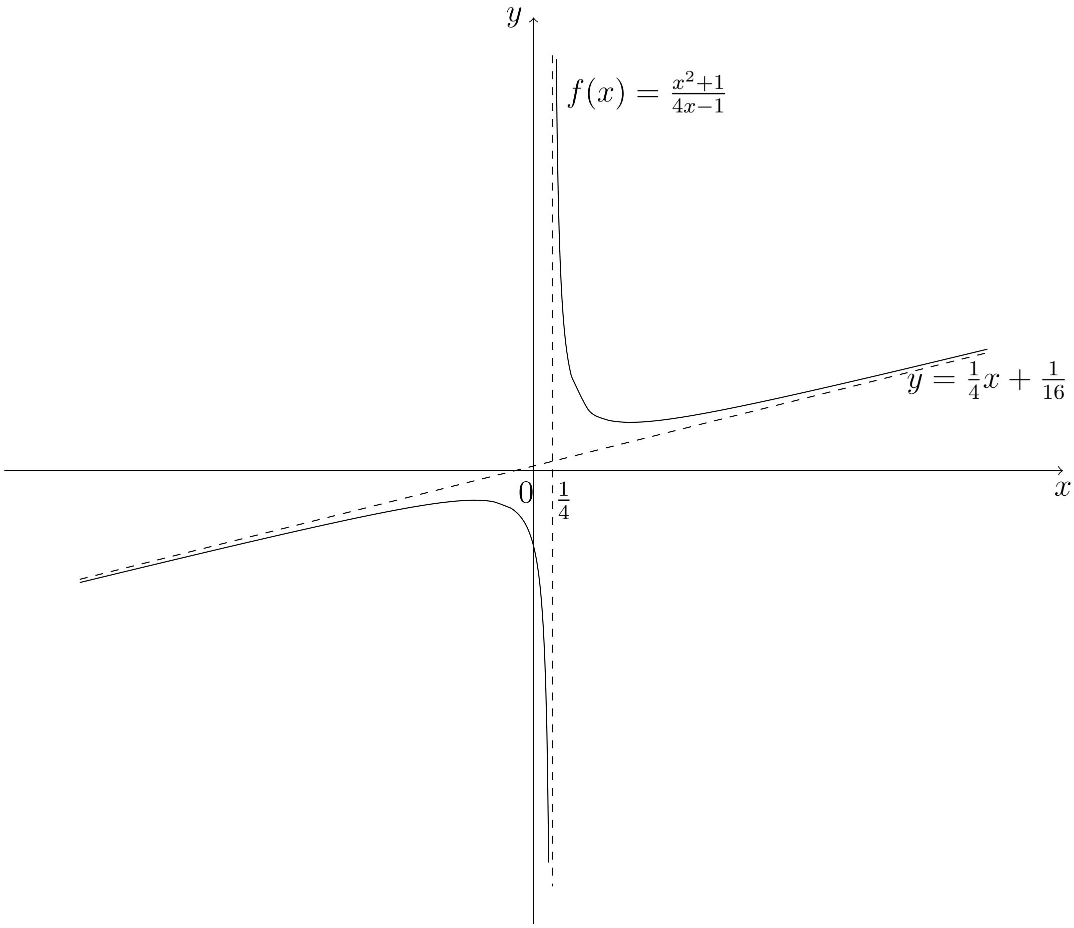

Figure 19