Quadratic function

Definition 1 *If a, b and c are real numbers with $a \neq 0$, then the function*

$$f(x) = ax^2 + bx + c$$

is a quadratic function and its graph is a parabola.

Any quadratic function can be transformed into the vertex form:

$$f(x) = a(x-h)^2 + k,$$

where the point $V(h;k)$ is the vertex of the parabola. This process is called the completing the square.

Example 1 *Find the vertex of the parabola $f(x) = 5x^2 - 2x + 8$.*

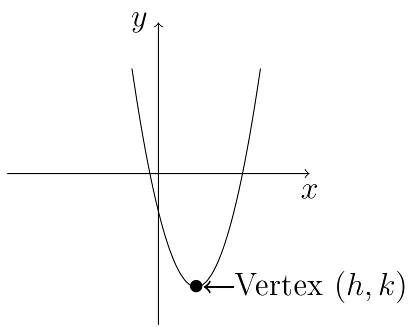
Solution 1

$$f(x) = 5\left(x^2 - \frac{2}{5}\cdot x\right) + 8$$

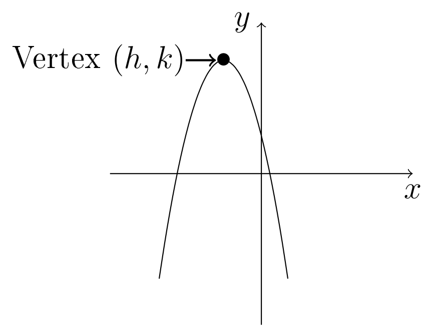
$$= 5\left(x^2 - 2\cdot\frac{1}{5}\cdot x + \frac{1}{25}\right) - \frac{1}{5} + 8 = 5\left(x - \frac{1}{5}\right)^2 + \frac{39}{5}.$$

Thus, the point $V\left(\frac{1}{5};\frac{39}{5}\right)$ is the vertex of the parabola $f(x) = 5x^2 - 2x + 8$.

If $a > 0$, then the graph of a quadratic function $f(x) = ax^2 + bx + c$ opens upward. If $a < 0$, then the graph of a quadratic function $f(x) = ax^2 + bx + c$ opens downward.



(A) $a > 0$



(B) $a < 0$

Figure 1

Applications

Let x be a number of units manufactured and sold. It will represent the independent variable. Let a, b, m and n be real numbers. Then we can define the following functions:

1. cost function

$$C(x) = (\text{variable cost}) + (\text{fixed cost}) = ax + b;$$

2. price-demand function (suppose that all produced goods can be sold at a price P per unit)

$$P(x) = mx + n;$$

3. revenue function

$$R(x) = (\text{number of units sold}) \cdot (\text{price per unit}) = x \cdot P(x) = x \cdot (mx + n);$$

4. profit function

$$P(x) = R(x) - C(x) = x \cdot (mx + n) - (ax + b).$$

A loss occurs if $R(x) < C(x)$ and a profit occurs if $R(x) > C(x)$. Break-even points are the production levels at which $R(x) = C(x)$.

Example 2 *A company that produces watches has fixed costs (at* 0 *output) of \$12000 per month; and variable costs of \$300 per watch. The price is \$500. Find the break-even point.*

Solution 2 *The cost function is written as follows* $C(x) = 300 \cdot x + 12000$. *The price is fixed, so the price-demand function is* $P(x) = 500$. *The revenue function has the form* $R(x) = 500 \cdot x$. *To find the break-even point, we need to solve the equation* $R(x) = C(x)$. *Thus,*

$$500 \cdot x = 300 \cdot x + 12000$$

$$200 \cdot x = 12000$$

$$x = 60 \ \ \textit{watches}.$$

Moreover,

$$R(60) = C(60) = 500 \cdot 60 = \$\ 30000.$$

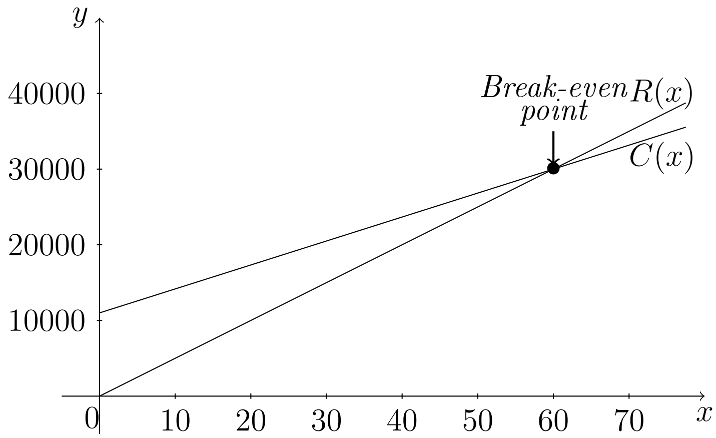


Figure 2

Polynomial function

Definition 2 *A polynomial function is a function that can be written in the form:*

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

for a non-negative integer n called the degree of the polynomial. The coefficients a_0, a_1, ..., a_{n-1} and a_n are real numbers with $a_n \neq 0$.

The domain of a polynomial function is the set of all real numbers.

Rational function

Definition 3 *A rational function is a function that can be written in the form:*

$$f(x) = \frac{n(x)}{d(x)},$$

where $n(x)$ and $d(x)$ are polynomials.

The domain of a rational function is the set of all real numbers such that $d(x) \neq 0$. We assume that $n(x)$ and $d(x)$ have no real zeroes in common.

Exponential function

Definition 4 *The equation*

$$f(x) = b^x, \quad b > 0, \quad b \neq 1,$$

defines an exponential function for each different constant b, called the base.

The domain of an exponential function $f(x) = b^x$ is the set of all real numbers. The range is the set of all positive real numbers.

Basic properties of graphs of exponential functions

1. All graphs pass through the point $(0;1)$, since $b^0 = 1$ for any permissable base b.
2. All graphs are continuous curves with no holes and jumps.
3. If $b > 1$, then b^x increases as x increases.
4. If $0 < b < 1$, then b^x decreases as x increases.
5. The x-axis is a horizontal asymptote.

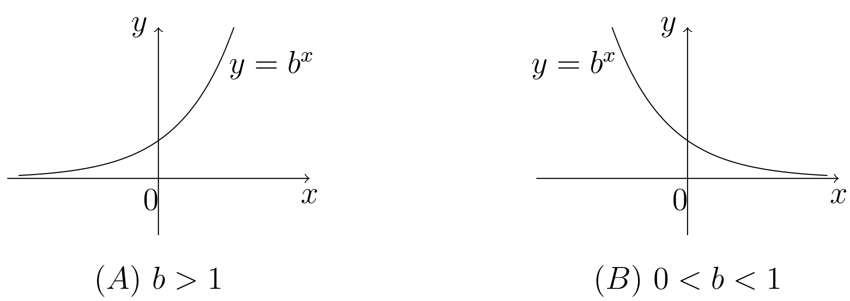


Figure 3

Base e exponential function

The number e is an irrational number that can be approximated by evaluating the expression

$$\left(1 + \frac{1}{x}\right)^x$$

for a sufficiently large x. It means that if x increases without bound, the value of the above expression approaches a number we denote e. The following table illustrates this fact:

x	1	10	100	1000	10000	100000	1000000
$\left(1+\frac{1}{x}\right)^x$	2	2.59374...	2.70481...	2.71692...	2.71814...	2.71827...	2.71828...

Therefore, the irrational number e is approximately equal to $e \approx 2.718...$

Exponential functions with base e and base $\frac{1}{e}$ are defined by $y = e^x$ and $y = e^{-x}$, respectively.

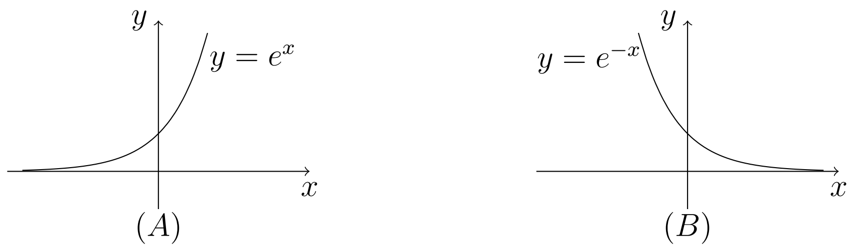


Figure 4

Properties of exponential functions

If a and b are positive real numbers such that $a \neq 1$ and $b \neq 1$, and x and y are real numbers, then

1. $a^x a^y = a^{x+y}$ and $\frac{a^x}{a^y} = a^{x-y}$;
2. $(a^x)^y = a^{xy}$, $(ab)^x = a^x b^x$ and $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$;
3. $a^x = a^y$ if and only if $x = y$;
4. for $x \neq 0$, $a^x = b^x$ if and only if $a = b$.

Logarithmic function

To define a logarithmic function, we need to introduce the definition of inverse function. Let us consider two functions: $f(x) = x$ and $g(x) = |x|$. By the definition of function, we know that each domain value corresponds to unique range value. The inverse statement is not always correct. In fact, if the range value of f is 2, the corresponding domain value is unique and equal to 2. However, if the range value of g is 2, there are two corresponding domain values 2 and -2. Function f is said to be an one-to-one function.

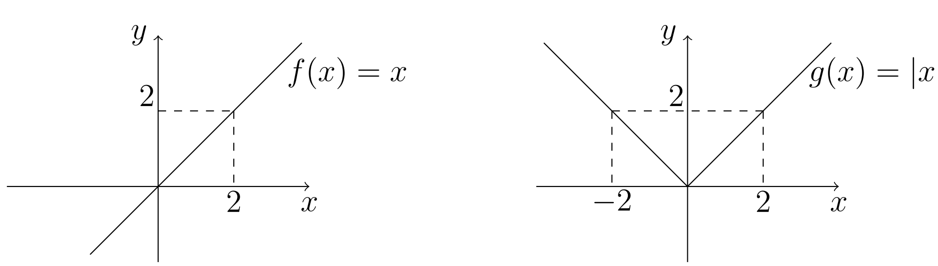


Figure 5

Definition 5 *A function f is said to be one-to-one if each range value corresponds to exactly one domain value.*

Definition 6 *If a function f is one-to-one, the inverse of f is the function formed by interchanging the independent and dependent variables.*

Remark 1 *If a function f is not one-to-one, it does not have an inverse.*

Remark 2 *If a function is one-to-one, then it is either increasing or decreasing for all domain values. If a function increases for some domain values and decreases for others, then it is not one-to-one.*

It is obvious that any exponential function is one-to-one, therefore, it has the inverse function.

Definition 7 *If we start with the exponential function*

$$y = b^x$$

and interchange the variables, we get the inverse

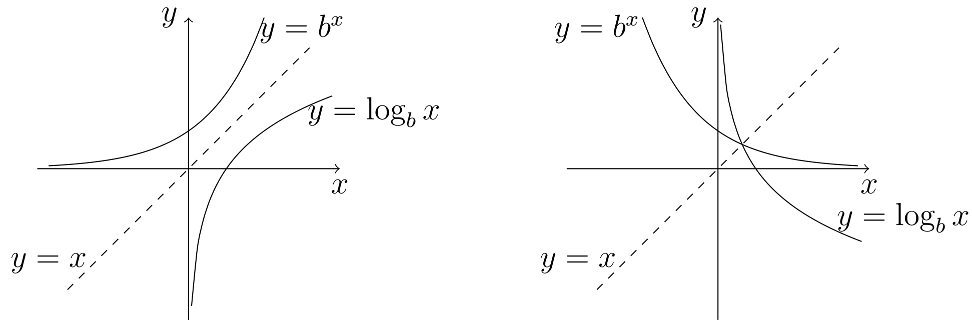
$$x = b^y.$$

We call the inverse the logarithmic function with base b and write

$$y = \log_b x.$$

The domain of a logarithmic function $y = \log_b x$ is the set of all positive real numbers. The range is the set of all real numbers. Moreover, all graphs pass through the point $(1; 0)$.

The graphs of $y = b^x$ and $y = \log_b x$ are shown in Figure 6 for the cases $b > 1$ and $0 < b < 1$.



$(A)\ b > 1$ $(B)\ 0 < b < 1$

Figure 6

Properties of logarithmic functions

If b, M and N are positive real numbers such that $b \neq 1$, and x and p are real numbers, then

1. $\log_b 1 = 0$;
2. $\log_b b = 1$;
3. $\log_b b^x = x$;
4. $b^{\log_b x} = x$ for $x > 0$;
5. $\log_b MN = \log_b M + \log_b N$;
6. $\log_b \frac{M}{N} = \log_b M - \log_b N$;
7. $\log_b M^p = p \log_b M$;
8. $\log_b M = \log_b N$ if and only if $M = N$.

Remark 3 *Common logarithm is a logarithm with the base* 10*:* $\log_{10} x = \log x$.

Natural logarithm is a logarithm with the base e*:* $\log_e x = \ln x$.

Applications

1. The fee paid to use another's money is called *interest*. It is usually presented as a percent and called *interest rate*. Suppose that you deposit P value of money at the annual interest rate $R\%$ (annual interest rate can be expressed as a decimal number r, so that $r = \frac{R}{100}$). After one year, the account balance is

$$P + R\% \text{ of } P = P + r \cdot P = P(1 + r).$$

At the end of the second year, the account balance is

$$(P(1+r))+R\% \text{ of } (P(1+r)) = P(1+r)+r\cdot P(1+r) = P(1+r)(1+r) = P(1+r)^2.$$

At the end of the third year, the account balance is

$$(P(1+r)^2)+R\% \text{ of } (P(1+r)^2) = P(1+r)^2+r\cdot P(1+r)^2 = P(1+r)^2(1+r) = P(1+r)^3.$$

If we continue this process, it is obvious that after t years, the account balance can be calculated using the formula:

$$A = P(1+r)^t,$$

where A is the amount at the end of t years.

Arguing similarly, we can write the general compound interest formula:

$$A = P\left(1+\frac{r}{m}\right)^{mt},$$

where

$A =$ future value or amount at the end of t years,

$P =$ present value or principle,

$r =$ annual rate, expressed as a decimal,

$m =$ number of compounding periods per year,

$t =$ time in years.

Example 3 *Suppose \$5000 is invested in an account paying 9% compounded monthly. How much will be in account after 5 years? 10 years?*

Solution 3

$$A = 5000\left(1+\frac{0.09}{12}\right)^{12t} = 5000\left(1+0.0075\right)^{12t} = 5000\left(1.0075\right)^{12t}.$$

If $t = 5$, *then*

$$5000\left(1.0075\right)^{12\cdot 5} \approx 5000\cdot 1.565681 \approx \$7828.41$$

If $t = 10$, *then*

$$5000\left(1.0075\right)^{12\cdot 10} \approx 5000\cdot 2.451357 \approx \$12256.79$$

2. Suppose we need to know the length of time it takes the value of an investment to double or *doubling time*. Thus, from the general compound interest formula we have

$$2P = P\left(1+\frac{r}{m}\right)^{mt}$$

$$2 = \left(1+\frac{r}{m}\right)^{mt}$$

$$\ln 2 = \ln\left(1+\frac{r}{m}\right)^{mt}$$

$$\ln 2 = m \cdot t \cdot \ln\left(1+\frac{r}{m}\right)$$

$$t = \frac{\ln 2}{m \cdot \ln\left(1+\frac{r}{m}\right)}$$

In particular, for rates compounded annually we have

$$t = \frac{\ln 2}{\ln(1+r)}$$

Example 4 *How long will it take money to double if it is invested at 20% compounded annually?*

Solution 4

$$t = \frac{\ln 2}{\ln(1+0.2)} = \frac{\ln 2}{\ln 1.2} = \frac{0.693147}{0.182322} \approx 3.801784 \approx 4 \ \ \textit{years.}$$

When interest is paid at the end of 4 years, the money will be slightly more than doubled.