6 Integration

6.1 Antiderivatives and indefinite integrals

Definition 1 *A function F is an antiderivative of a function $y = f(x)$ if $F'(x) = f(x)$.*

For example, $F(x) = \sin x$ is an antiderivative of $f(x) = \cos x$ since

$$(\sin x)' = \cos x.$$

If $F(x)$ is an antiderivative of $f(x)$, then $F(x)+C$, where C is a constant, is also an antiderivative of $f(x)$ since $(F(x) + C)' = f(x)$.

Definition 2 *We use the symbol*

$$\int f(x)dx$$

called the indefinite integral, to represent the family of all antiderivatives of $f(x)$, and write

$$\int f(x)dx = F(x) + C$$

if $F'(x) = f(x)$.

The symbol $\int$ is called the integral sign, and the function $f(x)$ is called the integrand. The dx in the integral indicates the variable x of integration.

Since finding of an antiderivative is the inverse operation to finding of a derivative, each rule for derivatives leads to a rule for antiderivatives. Thus, we write the table of main antiderivatives and their properties.

Table of main antiderivatives

1. $\int dx = x + C$;

2. $\int x^m dx = \frac{x^{m+1}}{m+1} + C$, $m \neq -1$;

3. $\int \frac{dx}{x} = \ln|x| + C$;

4. $\int a^x dx = \frac{a^x}{\ln a} + C$,

$\int e^x dx = e^x + C$;

5. $\int \sin x dx = -\cos x + C$;

6. $\int \cos x dx = \sin x + C$;

7. $\int \frac{dx}{\cos^2 x} = \tan x + C$;

8. $\int \frac{dx}{\sin^2 x} = -\cot x + C$;

9. $\int \frac{dx}{1+x^2} = \arctan x + C$;

10. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$.

Properties of antiderivatives

1. $\int (f(x) \pm g(x))\, dx = \int f(x)dx \pm \int g(x)dx$;

2. $\int k \cdot f(x)dx = k \int f(x)dx$, where k is a real number.

6.2 Methods of integration

There are three methods of integration: direct method, integration by substitution and integration by parts.

Direct method

This method is based on the table of main antiderivatives and their properties.

Example 1 *Evaluate the integral* $\int \left(\sqrt[3]{x} - 7e^x\right) dx$.

Solution 1

$$\int \left(\sqrt[3]{x} - 7e^x\right) dx = \int \sqrt[3]{x}dx - \int 7e^x dx = \int x^{\frac{1}{3}} dx - 7 \int e^x dx$$

$$= \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} - 7e^x + C = \frac{3x^{\frac{4}{3}}}{4} - 7e^x + C.$$

Integration by substitution

This method is based on the introduction of a new variable t such that $x = g(t)$ and $dx = dg(t) = g'(t)dt$. Thus,

$$\int f(x)dx = \int f(g(t)) \cdot g'(t)dt.$$

Remark 1 *In Remark* **??** *we point out that the notations* $f'(x)$ *and* $\frac{df(x)}{dx}$ *represent the derivative of* f*. Therefore,*

$$f'(x) = \frac{df(x)}{dx} \quad or \quad df(x) = f'(x)dx.$$

Similarly, we obtain the equality $dg(t) = g'(t)dt$ used in the substitution technique.

Example 2 *Evaluate the integral $\int x \cdot e^{4x^2-1}dx$.*

Solution 2 *We introduce the following substitution: $t = 4x^2 - 1$. Therefore, $dt = d(4x^2 - 1) = 8x \cdot dx$ that gives $x \cdot dx = \frac{1}{8}dt$. Thus,*

$$\int x \cdot e^{4x^2-1}dx = \int e^t \frac{1}{8}dt = \frac{1}{8}\int e^t dt = \frac{1}{8}e^t + C = \frac{1}{8}e^{4x^2-1} + C.$$

Integration by parts

This method is based on the product formula of derivatives. Suppose that $u(x)$ and $v(x)$ are differentiable functions over some interval (a, b). Then $(u \cdot v)' = u' \cdot v + v' \cdot u$. By Remark 1, we write

$$d(u \cdot v) = (u \cdot v)'dx$$

or

$$d(u \cdot v) = (u' \cdot v + v' \cdot u)dx = u' \cdot v \cdot dx + v' \cdot u \cdot dx = v \cdot u' \cdot dx + u \cdot v' \cdot dx = v \cdot du + u \cdot dv.$$

Therefore,

$$\int d(u \cdot v) = \int (v \cdot du + u \cdot dv) = \int v \cdot du + \int u \cdot dv$$

$$u \cdot v = \int v \cdot du + \int u \cdot dv$$

$$\int u \cdot dv = u \cdot v - \int v \cdot du.$$

The last formula is known as the integration by parts.

Selection of u and dv

Recommendations for selecting u and dv for some types of integrals are given below. Suppose that $P(x)$ is a polynomial and a and b are real numbers.

1. For integrals $\int P(x) \cdot e^{ax}dx$, $\int P(x) \cdot \sin bx dx$ and $\int P(x) \cdot \cos bx dx$, try $u = P(x)$ and dv is the rest;

2. For integrals $\int P(x) \cdot \ln x dx$, $\int P(x) \cdot \arcsin x dx$, $\int P(x) \cdot \arccos x dx$ and $\int P(x) \cdot \arctan x dx$, try $dv = P(x)dx$ and u is the rest;

3. For integrals $\int e^{ax} \cdot \sin bx dx$ and $\int e^{ax} \cdot \cos bx dx$, try $u = e^{ax}$ and dv is the rest.

Example 3 *Evaluate the integral* $\int (2x+1)\cdot e^{3x}dx$.

Solution 3 *Let*

$$u = 2x + 1 \quad \textit{and} \quad dv = e^{3x}dx.$$

Then

$$du = 2dx \quad \textit{and} \quad v = \int e^{3x}dx = \frac{1}{3}e^{3x}.$$

Substitute the results into the integration by parts formula:

$$\int (2x+1)\cdot e^{3x}dx = (2x+1)\cdot \frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x}\cdot 2dx =$$

$$\frac{1}{3}(2x+1)e^{3x} - \frac{2}{3}\int e^{3x}dx = \frac{1}{3}(2x+1)e^{3x} - \frac{2}{9}e^{3x} + C.$$